# Isogonal conjugate and a few properties of the point $X_{25}$

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**Abstract**

In this article I'm defining hodpiece of a point in a triangle and I study its properties. Some additional properties of $X_{25}$ (a special case of a hodpiece) are presented, and I also prove that the fixed point of this operation is impossible to construct using only straightedge and compass.

## 1 Introduction

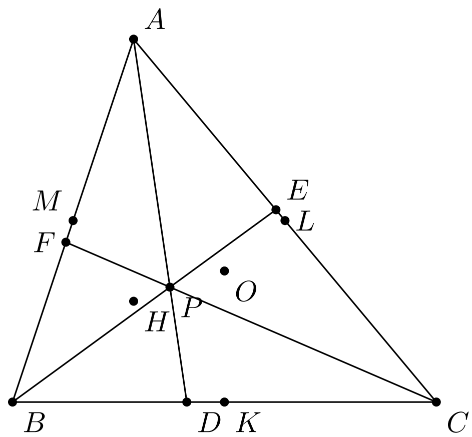
Isogonal conjugate of a line is a conic – this simple fact is the foundation of this article. On this base I define hodpiece[1] of a point in a given triangle. The third section contains the proof of its existence. Then I examine its properties in particular cases. Considering our construction for medial lines of a triangle we obtain the point $X_{25}$, whose some of the other properties are shown in the fourth chapter. Hodpieces of points on the circumcircle turns out to be the centroid of the triangle. At the end I prove that there exists a unique point which is its own hodpiece, and that it's impossible to construct it in a classical way, that is with straightedge and compass.

[1] I have used physicians' strategy to steal neologisms from Joyce if necessary[2]

[2] "has the most conical hodpiece of confusianist heronim", *Finnegans Wake*, James Joyce, page 131

# 2 Preliminaries

## 2.1 Notation




Let $ABC$ be a triangle. We introduce following notation:

- $K$, $L$, $M$ – midpoints of sides
- $P$ – any point
- $D$, $E$, $F$ – intersections of lines $AP$, $BP$, $CP$ with opposite sides of a triangle
- $O$ – circumcenter of $ABC$
- $H$ – orthocenter of $ABC$

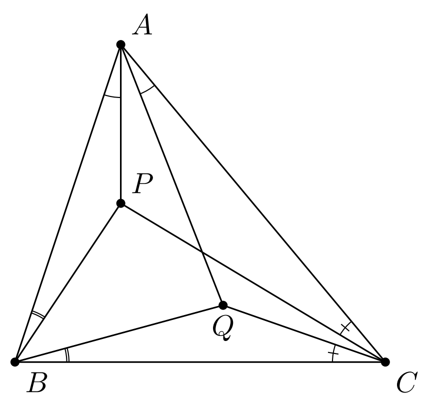
$\sphericalangle XYZ$ will denote directed angle between $YX$ and $YZ$.

**Warning.** In the fourth chapter some points will have other meaning, respective notation will be given at the beginning of that chapter.

In a given triangle it's possible to define many different triangle centers (e.g. centroid or orthocenter). Such points are gathered in Encyclopedia of Triangle Centers [2]; we will denote $n$-th of those centers as $X_n$.

## 2.2 Isogonal conjugate

We will say that points $P$ and $Q$ are *isogonally conjugated* in triangle $ABC$, if $\sphericalangle BAP = \sphericalangle QAC$, $\sphericalangle CBP = \sphericalangle QBA$ and $\sphericalangle ACP = \sphericalangle QCB$. Of course this is a symmetric relation.




**Theorem 2.1.** If $P$ does not lie on lines containing sides of the triangle, it has an isogonal conjugate.

*Proof.* Let $P_A$, $P_B$, $P_C$ be reflections of $P$ with respect to sides of the triangle. Denote circumcenter of $P_AP_BP_C$ as $Q$. Then $P_CA = P_BA$ $(= PA)$ and $QP_C = QP_B$, so triangles $P_CAQ$ and $QAP_B$ are congruent. In particular $\sphericalangle P_CAQ = \sphericalangle QAP_B$. Since

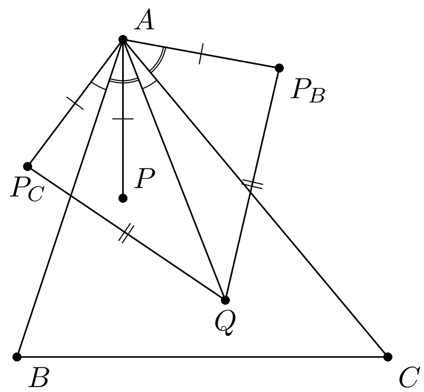
$$2\sphericalangle BAP + \sphericalangle PAQ = \sphericalangle P_CAB + \sphericalangle BAP + \sphericalangle PAQ = \sphericalangle P_CAQ$$

and

$$2\sphericalangle QAC + \sphericalangle PAQ = \sphericalangle QAC + \sphericalangle PAC = \sphericalangle QAC + \sphericalangle CAP_B = \sphericalangle QAP_B$$

we finally obtain $\sphericalangle BAP = \sphericalangle QAC$. We derive two remaining equalities in a similar way.

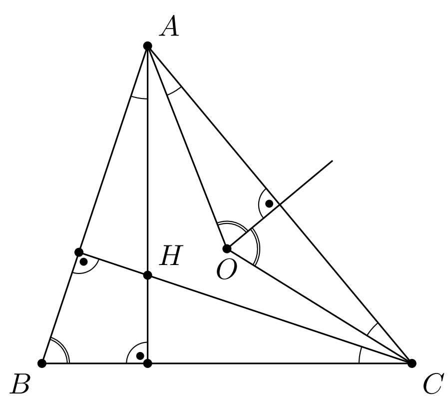
$\square$




**Warning.** If points $P_A$, $P_B$, $P_C$ are collinear (which happens exactly when $P$ lies on a circumcircle of $ABC$[3]), point $Q$ will be direction orthogonal to line passing through them. Then both $A$ and $Q$ lie on perpendicular bisector of $P_BP_C$, so equality $\sphericalangle P_CAQ = \sphericalangle QAP_B$ is true. The rest of reasoning stays the same.

Isogonal conjugate of $P$ will be denoted by $P^*$.

**Example.** $\sphericalangle BAH = 90^\circ - \sphericalangle CBA = 90^\circ - \frac{1}{2}\sphericalangle COA = \sphericalangle OAC$, so $H$ and $O$ are isogonal conjugates of each other.

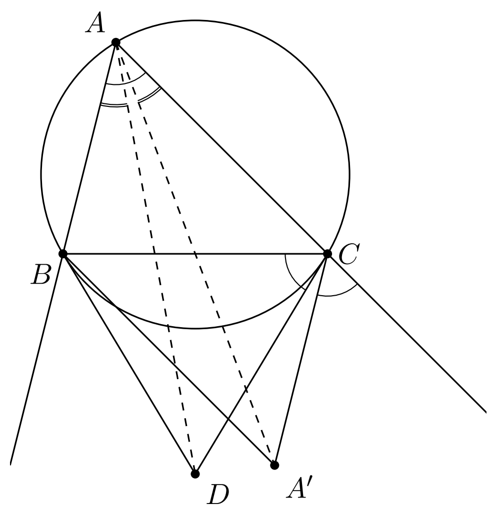



**Example.** Let $M$ be the midpoint of side $BC$. Reflect $A$ through $M$ obtaining $A'$. Moreover let tangents to circumcircle in $B$ and $C$ intersect in $D$. Then

$$-\sphericalangle A'CA = \sphericalangle BAC = \sphericalangle BCD$$

[3]It will be the Steiner line of $P$

and in a similar way we find analogous equality for angles with vertex $B$, so $A'$ and $D$ are isogonally conjugate. In particular $AD$ is a symmedian (i.e. line isogonally conjugate to median line) in angle $BAC$.




## 2.3 Barycentric coordinates

If for some $P$ and real numbers $p_A$, $p_B$, $p_C$ (not all equal to zero) identity

$$p_A\overrightarrow{PA} + p_B\overrightarrow{PB} + p_C\overrightarrow{PC} = \overrightarrow{0}$$

holds, we say that $[p_A : p_B : p_C]$ are barycentric coordinates of $P$ in $ABC$. Of course for any nonzero $\lambda$ if $[p_A : p_B : p_C]$ are barycentric coordinates of $P$, then so are $[\lambda p_A : \lambda p_B : \lambda p_C]$. Any point $P$ has some barycentric coordinates, unique up to constant, since $\overrightarrow{PA}$, $\overrightarrow{PB}$, $\overrightarrow{PC}$ span a space of dimension two, which means that kernel of map $\mathbb{R}^3 \ni (u, v, w) \mapsto u\overrightarrow{PA} + v\overrightarrow{PB} + w\overrightarrow{PC}$ has dimension equal to one. Barycentric coordinates can be interpreted as such masses $p_A$, $p_B$, $p_C$, that after putting them in $A$, $B$, $C$ we will get system of objects with center of mass in $P$; this shows that these coordinates uniquely determine a point, so we will denote $P \sim [p_A : p_B : p_C]$ if $[p_A : p_B : p_C]$ are barycentric coordinates of $P$. This also implies that barycentric coordinates of any point on a line $PQ$ are linear combination of coordinates of $P$ and $Q$.

In this interpretation we supposed $p_A + p_B + p_C \neq 0$, since only then respective system will have a center of mass. It turns out that points with sum of coordinates equal to zero are points in infinity.

Barycentric coordinates are especially useful in problems concerning triangle geometry, since they allow us to describe objects connected to it in an easy way. For example, $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ are triangle's vertices $A$, $B$, $C$, while $[1:1:1]$ is the centroid of $ABC$.

**Lemma 2.1.** Let $P \sim [p_A : p_B : p_C]$, and let lines $AP$, $BC$ intersect in $D$. Then

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} = \frac{p_C}{p_B}$$

*Proof.* Observe $D \sim [0 : p_B : p_C]$ – indeed, this coordinates are linear combinations of both coordinates of $A$ and $D$, and also of $B$ and $C$. From the definition

$$p_B\overrightarrow{DB} + p_C\overrightarrow{DC} = 0$$

which is equivalent to thesis. □

This observation is enough to prove both Ceva's and Menelaus's theorem. It shows equivalency of barycentric coordinates and segments' proportions, which we will be often using.

**Lemma 2.2.** Isogonal conjugate of point $P \sim [p_A : p_B : p_C]$ has barycentric coordinates

$$\left[\frac{a^2}{p_A} : \frac{b^2}{p_B} : \frac{c^2}{p_C}\right]$$

*Proof.* Let $Q = P^*$ and let $AP$, $AQ$ intersect $BC$ respectively in $D$, $E$. We want to prove

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \frac{c^2}{b^2}$$

Let $\theta = \sphericalangle BAD$. From law of sines for $BAD$ and $DAC$ we have

$$\frac{BD}{\sin\theta} = \frac{c}{\sin\sphericalangle ADB}$$

and

$$\frac{CD}{\sin(\alpha - \theta)} = \frac{b}{\sin\sphericalangle CDA}$$

Of course $\sin\sphericalangle ADB = \sin\sphericalangle CDA$, so dividing first equation by the second one, we get

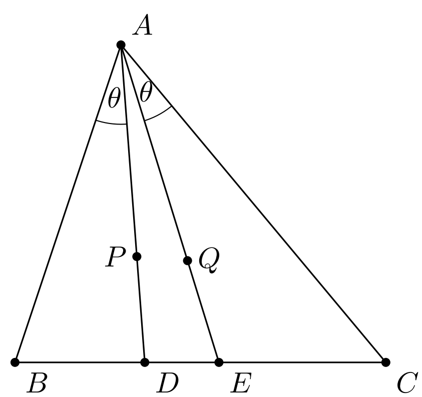
$$\frac{BD}{CD} \cdot \frac{\sin(\alpha-\theta)}{\sin\theta} = \frac{c}{b}$$

Analogously

$$\frac{BE}{CE} \cdot \frac{\sin\theta}{\sin(\alpha-\theta)} = \frac{c}{b}$$

Multiplying both equations yields thesis.

□




Thus a line described by the equation $\kappa x_A + \lambda x_B + \mu x_C = 0$ has isogonal conjugate with equation

$$\frac{\kappa a^2}{x_A} + \frac{\lambda b^2}{x_B} + \frac{\mu c^2}{x_C} = 0$$

which after multiplying by $x_A x_B x_C$ becomes

$$\kappa' x_B x_C + \lambda' x_C x_A + \mu' x_A x_B = 0$$

for some $\kappa'$, $\lambda'$, $\mu'$. Thus isogonal conjugate of a line is a curve of degree two – in barycentric coordinates, so in order to show this is a conic, we need the following lemma:

**Lemma 2.3.** Degree of an algebraic curve in barycentric coordinates is the same as in Cartesian coordinates.

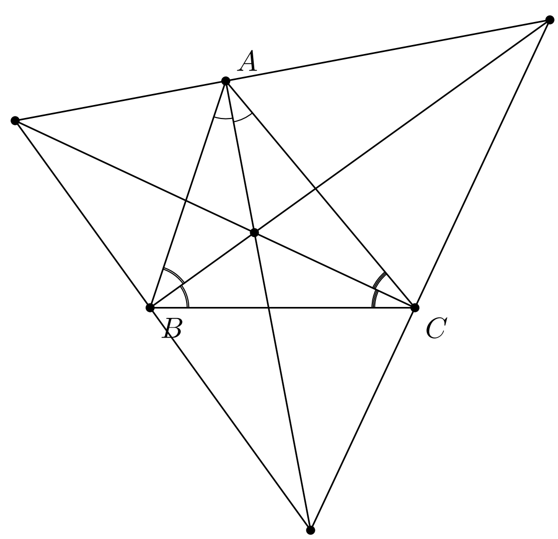
*Proof.* Take $A' = (1,0)$, $B' = (0,1)$, $C' = (0,0)$. Observe that point $P = (x,y)$ in Cartesian coordinates has barycentric coordinates $[x : y : 1-x-y]$ in triangle $A'B'C'$, since

$$\overrightarrow{AP}\cdot x+\overrightarrow{BP}\cdot y+\overrightarrow{CP}\cdot(1-x-y) = \overrightarrow{(x-1,y)}\cdot x+\overrightarrow{(x,y-1)}\cdot y+\overrightarrow{(x,y)}\cdot(1-x-y) = 0$$

Let $f(x,y)$ will be equation of the curve in Cartesian coordinates, while $G(x,y,z)$ in barycentrics. Then there exists such homogeneous polynomial $F(x,y,z)$ that $F(x,y,1-x-y) = f(x,y)$ (we multiply each monomial of lower degree in $f$ by $x+y+z$ in an appropriate power), and affine transformation mapping $A'B'C'$ onto $ABC$ will turn $F$ into $G$, thus since affine transformations are linear, in particular we have $\deg G = \deg F = \deg f$. $\square$

Therefore isogonal conjugate of a line is a conic passing through vertices of the triangle.

**Example.** We know that incenter is its own isogonal conjugate. Thus it must have property that $x_A = \frac{a^2}{x_A}$, so $x_A = a$ and coordinates of this point are $[a:b:c]$. Excenters too have this property (but they are lying outside of the triangle), so they will have additional minus on respective coordinate.




**Example.** Isogonal conjugate of line at infinity is the circumcircle. Since equation of this line is $x_A + x_B + x_C = 0$, equation of circumcircle of $ABC$ is

$$a^2 x_B x_C + b^2 x_C x_A + c^2 x_A x_B = 0$$

**Example.** We will prove that any hyperbola passing through $A$, $B$, $C$, $H$ is rectangular (ie. its asymptotes are perpendicular).

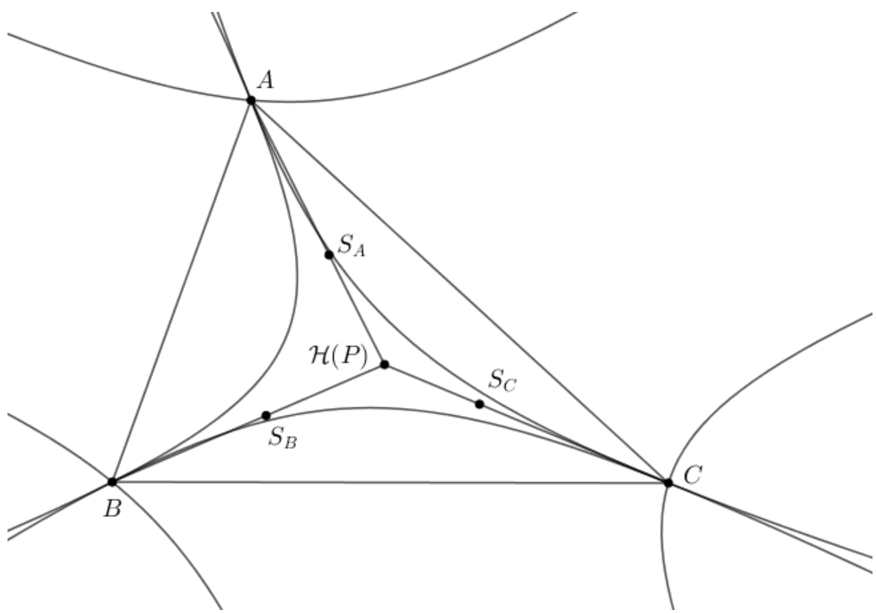
Its isogonal conjugate is a line passing through $O$. Let it intersect circumcircle in $X$ and $Y$. Then $X^*$ and $Y^*$ are directions of this hyperbola's asymptotes. Furthermore, $\sphericalangle XAY$ is right, as an angle subtended on a diameter. Therefore $\sphericalangle XAY = \sphericalangle X^*AY^* = 90°$, and this hyperbola is indeed rectangular.

## 3 Main construction

Choose a point $P$. Let $AP$, $BP$, $CP$ intersect opposite sides in $D$, $E$, $F$. Isogonal conjugate of line $EF$ is some conic; let $S_A$ be its center. We define $S_B$ and $S_C$ in a similar way.

**Theorem 3.1.** Lines $AS_A$, $BS_B$, $CS_C$ concur.

**Definition.** We call this intersection the *hodpiece* of point $P$, denoted by $\mathcal{H}(P)$.




We will divide the proof in several steps:

**Lemma 3.1.** There are four points $A$, $B$, $C$, $D$ given in the plane, where $D$ lies inside triangle $ABC$ (in particular no four of these points are collinear).

Then there exists affine transformation $f$ such that $f(D)$ is orthocenter of triangles with vertices $f(A)$, $f(B)$, $f(C)$.

*Proof.* Let $D$ have barycentric coordinates $[d_A : d_B : d_C]$ in $ABC$. These coordinates are invariant in $f$, since affine transformations preserve proportions. Thus we want to transform $ABC$ into a triangle with such angles $\alpha$, $\beta$, $\gamma$ that

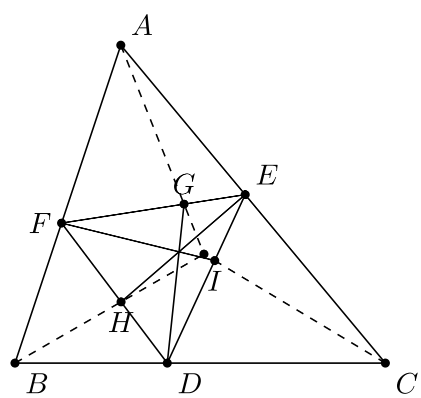
$$[d_A : d_B : d_C] = [\tan\alpha : \tan\beta : \tan\gamma],$$

since these are exactly barycentric coordinates of the orthocenter. Thus we want for some $\lambda$ that

$$\begin{cases} \alpha = \arctan \lambda d_A \\ \beta = \arctan \lambda d_B \\ \gamma = \arctan \lambda d_C \\ \alpha + \beta + \gamma = \pi \end{cases}$$

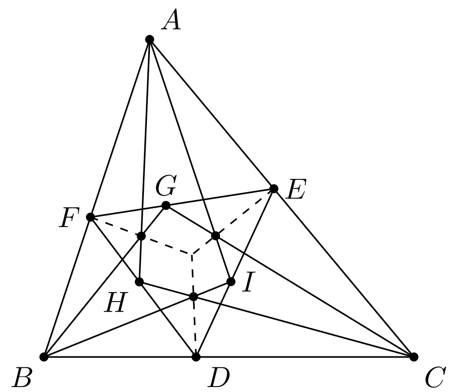
Observe that for $\lambda = 0$ also $\alpha + \beta + \gamma = 0$, while for $\lambda \to \infty$ this sum tends to $\frac{3}{2}\pi$. Therefore for some $\lambda$ we will get exactly $\pi$. $\square$

**Lemma 3.2** (Ceva Nest)**.** Let points $D$, $E$, $F$ lie on respective sides of triangle $ABC$, while $G$, $H$, $I$ on sides of $DEF$ in such a way, that $AD$, $BE$, $CF$ concur, as well as $DG$, $EH$ and $FI$. Then also lines $AG$, $BH$, $CI$ concur.




*Proof.* Proof can be found in [3]. $\square$

**Lemma 3.3.** Let points from $A$ to $I$ be situated as in the previous lemma. Let $CH$ intersect with $BI$ in $X$, $AI$ with $CG$ in $Y$, and $BG$ with $AH$ in $Z$. Then lines $DX$, $EY$ and $FZ$ concur.




*Proof.* From Lemma 3.2 $AG$, $BH$ and $CI$ concur, denote the point of concurrence as $P$. Then $(P, X)$, $(H, I)$, $(B, C)$ are opposite pairs of vertices of complete quadrilateral, so from Dual Desargues Involution Theorem ([4]) for point $D$ there exists involution sending $DP \to DX$, $DH \to DI$, $DB \to DC$.

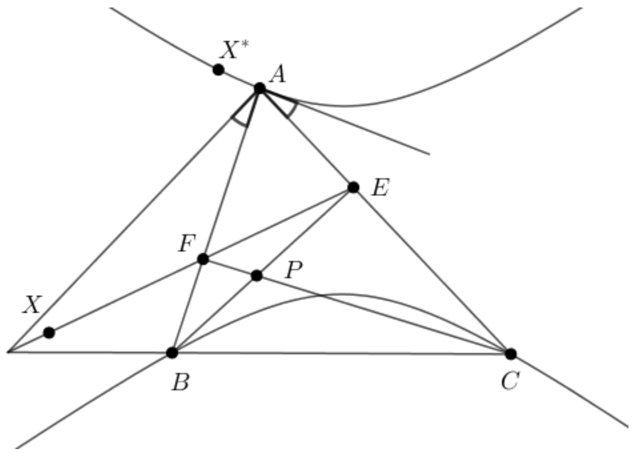
Denote intersection of $AD$, $BE$, $CF$ as $Q$. Since statement of the lemma is preserved in projective transformations, we may assume without loss of generality that $Q$ lies inside $ABC$, and from Lemma 3.1 after using appropriate affine transformation, we may suppose that $Q$ is orthocenter of $ABC$. Then from properties of orthic triangle line $DB$ is external bisector of angle $\sphericalangle FDE$, so involution from the last paragraph must be isogonal conjugation in $\sphericalangle FDE$ (since projective transformation is determined by its four (or even three) values, and we know that pairs $(DH, DI)$, $(DB, DC)$ are isogonally conjugated in this angle). Therefore $DP$ and $DX$ are also isogonally conjugated. Thus $DX$, $EY$, $FZ$ pass through isogonal conjugate of $P$ in $DEF$ (if $P$ lies on any side of the triangle, thesis is trivial, so we may assume that this isogonal conjugate indeed exists). $\square$

**Lemma 3.4.** For some points $Q_A$, $Q_B$, $Q_C$ such that points $A$, $B$, $C$ lie on sides of $Q_AQ_BQ_C$, we have $S_A = Q_BM \cap Q_CL$ (and symmetric equalities).

*Proof.* Denote isogonal conjugate of $EF$ as $\mathcal{S}_A$. Let $P_A$ (analogously $P_B$, $P_C$) will be such point, that $(P, P_A; A, D) = -1$, and let $Q$, $Q_A$, $Q_B$, $Q_C$ be isogonal conjugates of respective points $P$. We want to prove, that $P_BP_C$, $EF$ and $BC$ concur: let $T$ be intersection of $EF$ with $BC$, then $(T, D; B, C) = -1$ so projecting from $A$ to $CP$ we have, that $P_C$ lies on $AT$, analogously for $P_B$.

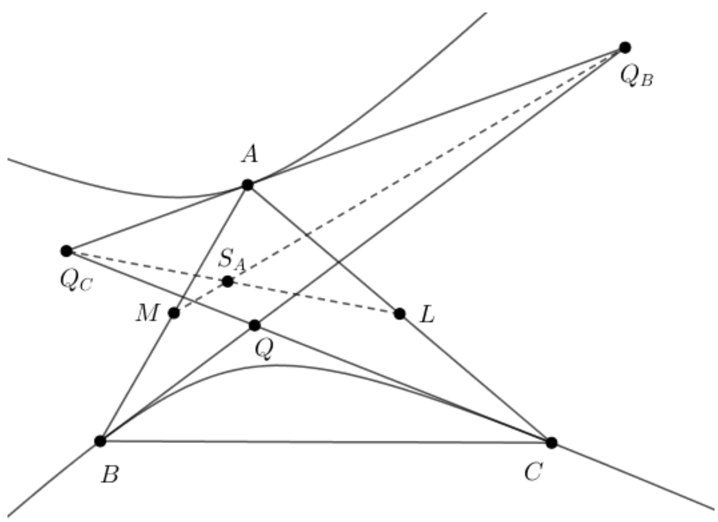
Let point $X$ move along the line $EF$ towards $T$. Then point $X^*$ moves along $\mathcal{S}_A$ towards $A$, where line $AX$ tends to $P_BP_C$, so $AX^*$ to $Q_BQ_C$. Thus

$Q_BQ_C$ is tangent in $A$ to $\mathcal{S}_A$. So are $BQ$ and $CQ$. This means that polar lines of points $Q$, $Q_B$, $Q_C$ are respectively $BC$, $AB$, $AC$.




Consider point in infinity of line $AC$ and denote it as $W$. Its polar line on one hand must pass through $L$, since $(W, L; A, C) = -1$, on the other hand it has to pass through pole of line $AC$, which is $Q_C$. Moreover pole of line of infinity is the center of conic, ie. $S_A$. Using similar argument for line $AB$ we get that $S_A$ is the intersection of $Q_BM$ with $Q_CL$. Using Lemma 3.3 concludes the proof of Theorem 3.1.

$\square$



We may also describe barycentric coordinates of $\mathcal{H}(P)$ in relation with $P$.

### 3.1 Barycentric coordinates of hodpiece

Let $P \sim [p_A : p_B : p_C]$, $Q \sim [q_A : q_B : q_C]$. Then $Q_A \sim [-q_A : q_B : q_C]$, analogously $Q_B$ i $Q_C$. Denote $s = q_A + q_B + q_C$ and take $S'_A \sim [q_A s : q_B(s - 2q_C) : q_C(s - 2q_B)]$. Observe, that writing $L$ as $2q_C \cdot [q_A : 0 : q_A]$ and $Q_C$ as $(s - 2q_C)[q_A : q_B : -q_C]$, and then summing these coordinates, we will get

$$\begin{cases} (2q_C + s - 2q_C)q_A = sq_A \\ (s - 2q_C)q_B \\ 2q_A q_C - (s - 2q_C)q_C = q_C(2q_A + 2q_C - s) = q_C(s - 2q_B) \end{cases}$$

so exactly coordinates of $S'_A$. Therefore $S'_A$ lies on line $Q_C L$ and similarly on $Q_B M$, thus $S'_A = S_A$. Then point

$$\mathcal{H}(P) := \left[\frac{q_A}{s - 2q_A} : \frac{q_B}{s - 2q_B} : \frac{q_C}{s - 2q_C}\right]$$

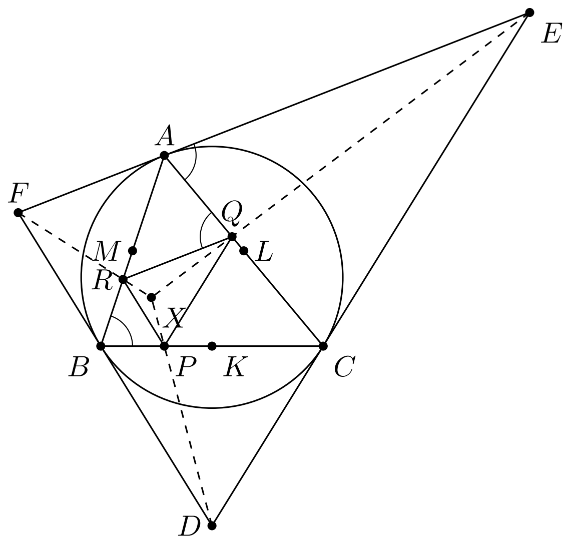
lies on lines $AS_A$, $BS_B$, $CS_C$, so this is $\mathcal{H}(P)$. Equivalently we can write it as

$$\left[\frac{a^2 p_B p_C}{b^2 p_C p_A + c^2 p_A p_B - a^2 p_B p_C} : \frac{b^2 p_C p_A}{c^2 p_A p_B + a^2 p_B p_C - b^2 p_C p_A} : \right.$$
$$\left. : \frac{c^2 p_C p_A}{c^2 p_A p_B + a^2 p_B p_C - b^2 p_C p_A}\right]$$

## 4 Center $X_{25}$

This section is devoted to the point we would get starting from medial lines of the triangle, ie. $\mathcal{H}(X_2)$. Theorem 4.1 proves this and other properties of this point.

Let $K$, $L$, $M$ be sides' midpoints, $P$, $Q$, $R$ foot of altitudes, and $D$, $E$, $F$ intersections of tangents to the circumcircle in $A$, $B$, $C$. Then $\sphericalangle AQR = \sphericalangle CBA = \sphericalangle CAE$, so $RQ \parallel EF$. Therefore respective sides of triangles $DEF$ and $PQR$ are parallel, so there exists homothety transforming one of them to another.




**Definition.** Center of this homotethy, which is intersection of lines $DP$, $EQ$ and $FR$, is a triangle center $X_{25}$ (in this chapter we will denote it as $X$ for simplifying notation).

## 4.1 Properties

Denote center of the circumcircle as $O$, while intersection of tangent to this circle in $A$ (ie. $DE$) with $BC$ as $T$.

**Theorem 4.1.** The following properties hold:

1. $FQ$, $RE$, $AO$ are concurrent.
2. $FL$, $EM$, $AX$ are concurrent.
3. Quadrilateral $ELMF$ can be inscribed in circle with center lying on $AX$.
4. Projection of $T$ onto $AX$ lies on the circumcircle.
5. Isogonal conjugate of $X$ is isotomic conjugate of $H$.
6. $X$ lies on Euler line.

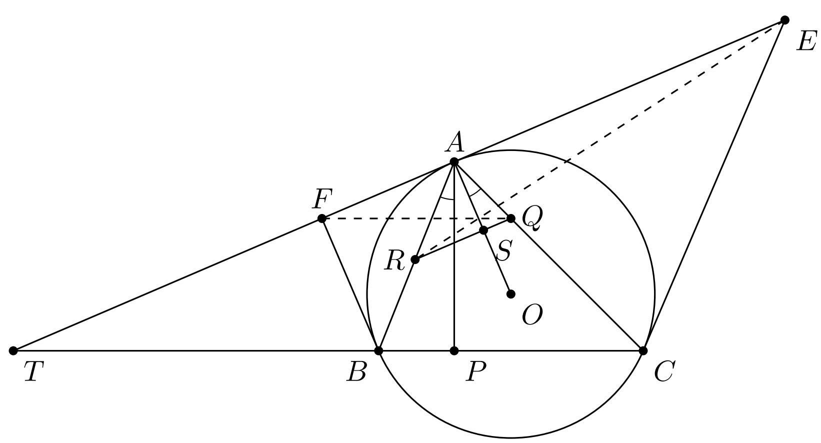
*Proof.* 1. Let $S$ be intersection of $AO$ with $RQ$. Since $RQ \parallel EF$, it suffices to prove that $\frac{FA}{EA} = \frac{QS}{RS}$. $AO \perp EF$, therefore also $AO \perp RQ$

and $S$ is a foot of altitude in triangle $RAQ$. Since triangles $CAB$ and $RAQ$ are similar, $\frac{QS}{RS} = \frac{BP}{CP}$. If we proved that triangles $FPB$ and $EPC$ are similar, we will get

$$\frac{AF}{AE} = \frac{BF}{CE} = \frac{BP}{CP} = \frac{QS}{RS},$$

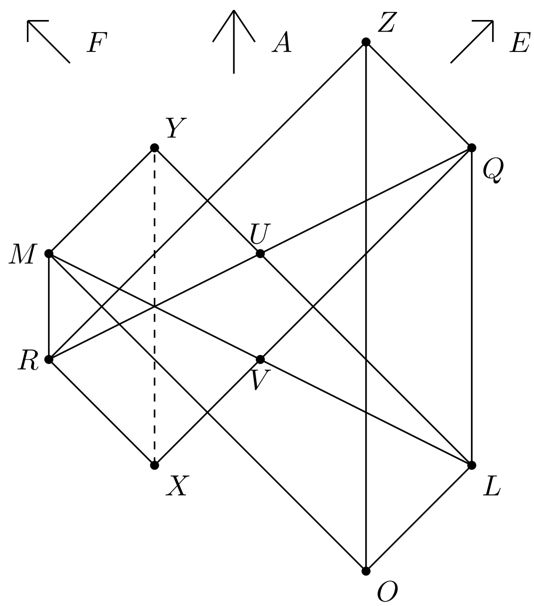
which we wanted to prove.

Observe that $\sphericalangle PBF = 180° - \sphericalangle DBC = 180° - \sphericalangle BCD = \sphericalangle ECP$. Therefore it's enough to prove that $PT$ is external bisector of angle $EPF$. This is indeed true – locus of such points $Z$, that $ZT$ is external bisector of angle $EZF$, is Apollonius circle with foci $E$, $F$ passing through $T$. Since $(A,T;E,F) = -1$, $AT$ has to be its diameter, while $\sphericalangle APT = 90°$, thus $P$ lies on this circle.



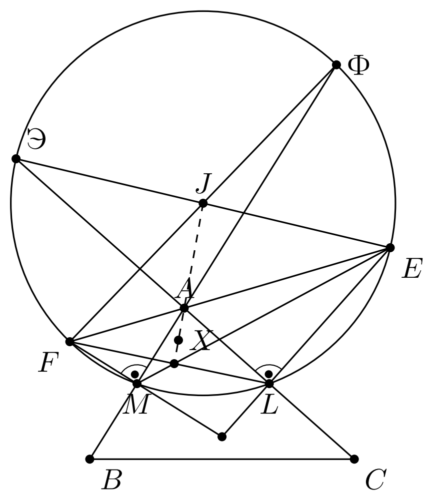

2. Denote intersection of $FL$ and $EM$ as $Y$, while of $FQ$ and $ER$ as $Z$. Observe, that defining $O$ as intersection of $EL$ with $FM$, we can ignore all information we have except for collinearities and concurrences which hold from definitions or first item of our theorem. Then without loss of generality we may assume $EF$ is the line at infinity. In this situation quadrilaterals $MOLY$ and $RXQZ$ are parallelograms, and lines $RM$, $LQ$, $OZ$ are parallel. Let $U$ be the center of segment $RQ$, while $V$ center of $LM$. If $U = V$, segment $XY$ is reflection of $ZO$ with respect to $V$, so it's parallel to this segment, which we wanted to prove. If it isn't the case, $UV$ is a medial line in trapezoid $RQLM$, thus it is

parallel to its base, and since it's also "medial line" in $XZOY$, and it's parallel to $OZ$, it must be also parallel to $XY$.[4]

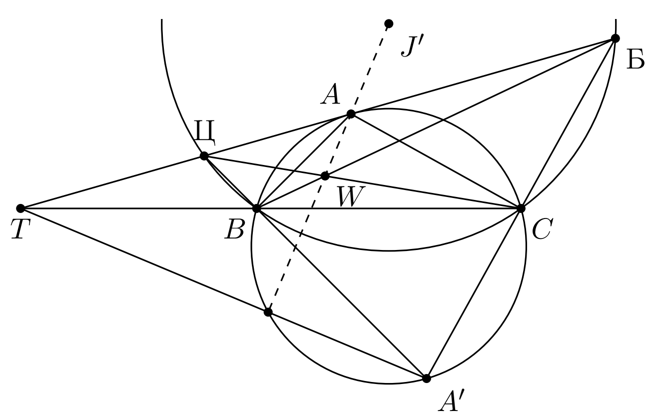



3. In inversion with respect to circumcircle $F$ is mapped to $M$, and $E$ to $L$, so $OF \cdot OM = OE \cdot OL = R^2$ and from power of a point theorem $EFML$ is inscribed in a circle. Let $J$ be its center. Reflect $E$ and $F$ with respect to $J$ to get Э and $\Phi$. Since $\sphericalangle AMF = 90°$, points $M$, $A$, $\Phi$ are collinear. Thus from Pascal theorem for hexagon $FL\text{Э}EM\Phi$ lines $AJ$, $FL$, $EM$ concur. But from item 2 of the theorem also $AX$, $FL$, $EM$ concur. Therefore $J$ lies on $AX$.

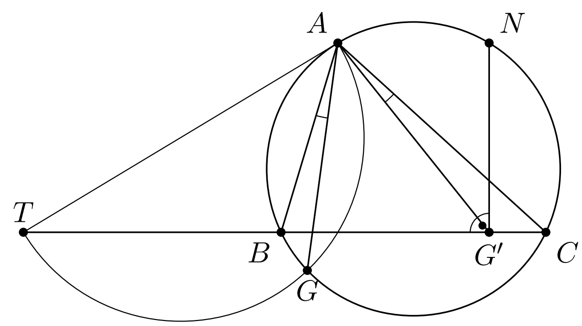
[4]Alternative proof: using Dual Desargues Involution Theorem for quadrilaterals with sides $(EQ, ER, FQ, FR)$ and $(EL, EM, FL, FM)$) for point $A$ respective involutions will be the same, and using part 1 of the theorem $AO$ is transformed to $AX$ in it.



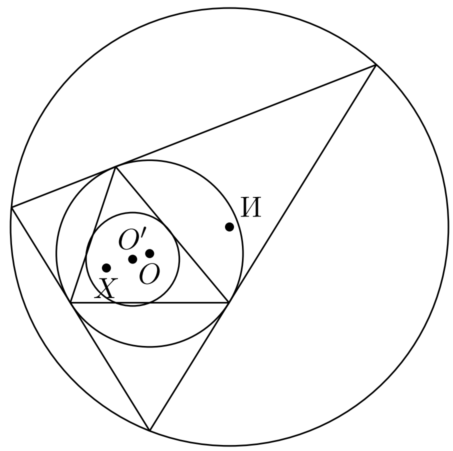

4. Reflect $A$ with respect to $E$ and $F$, to get Б and Ц. Let $C$Ц intersect $B$Б in $W$. Quadrilateral $BC$БЦ is the image of $MLEF$ in homothety with center $A$ and scale 2, so $W$ lies on $AX$ from item 2, moreover this quadrilateral is inscribed in circle $\Omega$, which center $J'$ lies on $AX$ (from item 3), and additionally $\measuredangle AB$Ц$=\measuredangle$Б$CA = 90°$. Let $A'$ be reflection of $A$ with respect to $O$. Then $B$Ц intersect $C$Б in $A'$, so from Brocard Theorem [5] $A'T$ is a polar line of $W$ with respect to $\Omega$. Therefore $J'W \perp A'T$, but $J'$, $A$, $W$ and $X$ are collinear, so $AX \perp A'T$. This means that intersection of these lines is at the same time projection of $T$ onto $AX$, and it lies on the circumcircle of $ABC$ (since $AA'$ is its diameter).



5. Let $N$ be such point that $ABCN$ is an isosceles trapezoid with base $BC$, while $G$ be projection of $T$ onto $AX$. Consider transformation $f$ being composition of inversion with center $A$ and radius $\sqrt{AB \cdot AC}$ with reflection across internal bisector of angle $BAC$. Then $f(B) = C$, $f(C) = B$, line $BC$ is mapped to circumcircle and *vice versa*. Moreover $\sphericalangle TAB = \sphericalangle ACB = \sphericalangle CAN$, so $f(T) = N$, and circle with diameter $AT$ is mapped to line through $N$ perpendicular to $BC$. Let $G' = f(G)$ and $X' = f(X)$. $G'$ is projection of $N$ onto $BC$, thus $BP = G'C$. Therefore $AG'$ (the same as $AX'$) is isotomic conjugate of $AH$, and isogonal conjugate of $AX$.




6. Denote center of nine point circle as $O'$, and circumcenter of $DEF$ as И. Since $X$ is a center of homothety transforming $PQR$ to $DEF$, $O'$ must be mapped to И, therefore these three points are collinear. Moreover $DEF$ is the image of $KLM$ in inversion with respect to circumcircle, so also $O$, $O'$, И are collinear. This means that $X$ lies on $O'O$, which is Euler line of $ABC$.




□

In particular from item 2 and proof of Theorem 3.1 follows that $\mathcal{H}(X_2) = X_{25}$, because $EF$ is isogonal conjugate of line parallel to $BC$ through $A$.

# 5 Other special cases

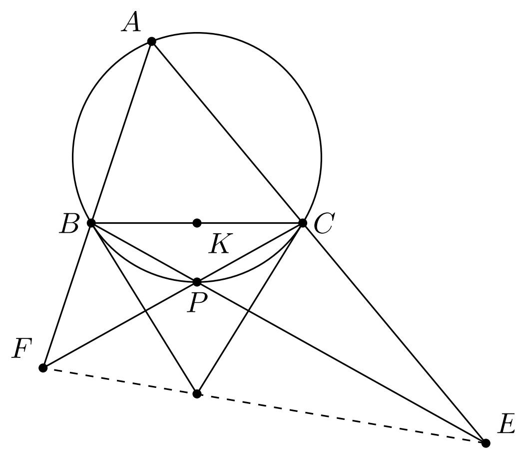
## 5.1 $P = X_1$

Case of $P$ being incenter was *de facto* considered inside the proof of Theorem 3.1, since we transformed statement of the problem to situation in which $Q$ jest is the incenter of $ABC$. We then proved that $\mathcal{H}(X_1)$ isogonal conjugate of intersection of lines $I_AK$, $I_BL$ and $I_CM$. Similarity $\triangle BI_AC \sim \triangle I_BI_AI_C$ shows that this intersection is Lemoine point of $I_AI_BI_C$, thus point $X_9$, and its isogonal conjugate is $X_{57}$, so $\mathcal{H}(X_1) = X_{57}$.

## 5.2 $P$ on the circumcircle

Since, as we've observed in Section 2, equation of the circumcircle is $a^2x_Bx_C + b^2x_Cx_A + c^2x_Ax_B = 0$, if $P$ lies on the circumcircle, $\mathcal{H}(P)$ is $X_2$. It can also be justified geometrically – if $Q = P^*$, tangents in $B$ and $C$ to isogonal conjugate of $EF$ pass through $Q$, therefore they are parallel – thus $K$ is the center of this conic.

That also means, that reflection of $A$ with respect to $K$ lies on this conic, and thence intersection of tangents in $B$ and $C$ to circumcircle lies on $EF$. In particular pole of $EF$ lies on $BC$. Since $P$ in this property has a role symmetric to that of $A$, $B$ and $C$, we can also say that pole of $EF$ lies on $AP$, which gives a proof of Brocard theorem without use of cross-ratio.




## 5.3 $P = X_n$

Comparing barycentric coordinates of first fifty triangle centers in ETC and obtained in Geogebra, we can obtain results as in the following table:

| $X_n$ | $\mathcal{H}(X_n)$ |
|---|---|
| 1 | 57 |
| 2 | 25 |
| 3 | 459 |
| 4 | 394 |
| 6 | 2 |
| 9 | 1422 |
| 19 | 6513 |
| 25 | 6340 |
| 31 | 6384 |

Lack of $\mathcal{H}(X_7)$ is particularly interesting, since in this case lines $DE$, $EF$, $FD$ has a more natural interpretation than for example for $P = X_3$.

## 5.4 Fixed point is unconstructible

Natural question about existence and number of solutions of $\mathcal{H}(P) = P$ is answered by the following theorem:

**Theorem 5.1.** For any triangle there exists exactly one such $P$ that $\mathcal{H}(P) = P$; this point cannot be obtained in a classical construction.

*Proof.* Let $P \sim [\frac{a^2}{x_A} : \frac{b^2}{x_B} : \frac{c^2}{x_C}]$. We want $P \sim [\frac{x_A}{x_B+x_C-x_A} : \frac{x_B}{x_C+x_A-x_B} : \frac{x_C}{x_A+x_B-x_C}]$, therefore

$$\frac{x_A^2}{a^2(x_B + x_C - x_A)} = \frac{x_B^2}{b^2(x_C + x_A - x_B)} = \frac{x_C^2}{c^2(x_A + x_B - x_C)}.$$

Observe that these expressions are homogenous and have nonzero degree, so by appriopriate scaling coordinates of $P$ we may assume that they are all equal to 1. Denote $y_A = \frac{x_A}{a}$, $y_B = \frac{x_B}{b}$, $y_C = \frac{x_C}{c}$. Then $y_A^2 = x_B + x_C - x_A$ i $y_B^2 = x_C + x_A - x_B$, thus $y_A^2 + y_B^2 = 2x_C = 2cy_c$. Denote $t = y_A^2 + y_B^2 + y_C^2$. Then

$$t + a^2 = y_A^2 + (y_B^2 + y_C^2) + a^2 = y_A^2 + 2ay_A + a^2 = (y_A + a)^2$$

therefore

$$\begin{cases} y_A = \sqrt{t + a^2} - a \\ y_B = \sqrt{t + b^2} - b \\ y_C = \sqrt{t + c^2} - c \\ t = y_A^2 + y_B^2 + y_C^2 = 3t + 2a^2 + 2b^2 + 2c^2 - 2a\sqrt{t + a^2} - 2b\sqrt{t + b^2} - 2c\sqrt{t + c^2} \end{cases}$$

Where the last expression after cancelations becomes

$$t + a^2 + b^2 + c^2 = a\sqrt{t + a^2} + b\sqrt{t + b^2} + c\sqrt{t + c^2}$$

Clearly $t$ as a sum of three squares, from which at least one is nonzero, has to be positive, so solution $t = 0$ doesn't correspond to any point $P$.

Observe that derivative of $a\sqrt{t + a^2}$ is $\frac{a}{2\sqrt{t+a^2}} \leqslant \frac{1}{2}$. Therefore if any of this roots was negative (which corresponds to $P$ lying outside the triangle), left hand side will have derivative equal to 1, while right hand side strictly less than 1. That means that left hand side will be negative for all positive $t$. However if all of the roots were positive, right hand side is a concave function, where its derivative in zero is greater than 1, so the line will intersect it in exactly one point (wherein that point exists because for large $t$ left hand side is greater than right).

It's left to justify nonconstrutibility. For any constructible numbers $a$, $b$ numbers $a + b$, $a - b$, $ab$, $\frac{a}{b}$ are also constructible (cases of sum and difference are obvious, remaining ones can be solved for example using power of point theorem). If $P$ was constructible, so would be some of its possible barycentric coordinates, for example

$$[p_A : p_B : p_C] = \left[1 : \frac{AF}{FB} : \frac{AE}{EC}\right]$$

(from Lemma 2.1 they are indeed correct). Therefore also coordinates of $Q = P^*$ are constructible, denote them as $[q_A : q_B : q_C]$. As we have proved, value of

$$\frac{q_A^2}{a^2(q_B + q_C - q_A)}$$

is independent of permutation of vertices, so we may divide coordinates by this value (which is constructible too), obtaining $x_A$ – in particular $x_A$, $x_B$, $x_C$ are constructible. From this we may easily construct $y_A$, $y_B$, $y_C$, and eventually $t$.

If alleged construction was working, in particular we could use it for triangle with $a = b = 2$, $c = 1$ (which is easy to construct if one have unit segment). Then $t$ satisfies equation

$$t + 9 = 4\sqrt{t+4} + \sqrt{t+1}$$

Squaring both sides

$$t^2 + 18t + 81 = 17t + 65 + 8\sqrt{(t+4)(t+1)}$$

Moving $17t + 65$ to the left hand side, and then squaring we get

$$t^4 + 2t^3 + 33t^2 + 32t + 256 = 64t^2 + 320t + 256$$

Moving to one side and using $t \neq 0$

$$t^3 + 2t^2 - 31t - 288 = 0$$

If $t$ is a constructible number, degree of its minimal polynomial must be a power of two from Wantzel's theorem. In particular above polynomial cannot be the minimal one of $t$, so it must be reducible. Of course such product has to contain linear factor, so this polynomial needs to have a rational root. From rational root theorem this root must be an integer divisor of 288. Moreover Sturm sequence of this polynomial is $P_0(x) = x^3+2x^2-31x-288$, $P_1(x) = P_0'(x) = 3x^2+4x-31$, $P_2(x) = \frac{1}{9}(194x+2530)$, $P_3(x) = -\frac{4018176}{9409}$, so from Sturm theorem for interval $(-\infty, \infty)$ it has exactly one real root. Since its value in 6 is $-186$, and in 8 is 104, this root must lie in interval $(6, 8)$, in particular it cannot be an integer divisor of 288. It gives a contradiction with constructibility of $P$.

$\square$

As far as author knows, this point hasn't been yet described in ETC. Its barycentric coordinates can be deduced from the proof to be equal to

$$\frac{a}{\sqrt{t+a^2}-a} : \frac{b}{\sqrt{t+b^2}-b} : \frac{c}{\sqrt{t+c^2}-c}$$

where $t$ is again the only positive solution of

$$t + a^2 + b^2 + c^2 = a\sqrt{t+a^2} + b\sqrt{t+b^2} + c\sqrt{t+c^2}$$

These coordinates are equivalent to

$$a^2 + a\sqrt{t+a^2} : b^2 + b\sqrt{t+b^2} : c^2 + c\sqrt{t+c^2}$$

## 6 Acknowledgments

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