A theorem by Dao Thanh Oai

Let ABC and A'B'C' be two not homothetic triangles. Let A"B"C" be any other triangle perspective to ABC and homothetic to A'B'C'. Then the perspectors of ABC and A"B"C" lie on a circumconic of ABC. (Dao Thanh Oai, March 17, 2018)

For building a such triangle A"B"C", let $Q = U : V : W$ (trilinears coordinates) be any point in the plane of ABC and q a line through Q. Denote A*, B*, C* the intersections of q with BC, CA and AB, respectively. Through A*, B*, C* trace parallel lines to the sidelines of A'B'C'. Then these parallel lines bound a triangle A"B"C" homothetic to A'B'C' and perspective to ABC with perspectrix q.

Let P be the perspector of ABC and A"B"C" and assume that the sidelines of A'B'C' are the lines $r_i = (l_i, m_i, n_i), i = 1..3$. Denote $R = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}$ and $\delta_{i,j} = (-1)^{i+j} \cdot Minor(R, i, j)$. Then the calculus shows that the locus of P when q rotates around Q has equation:

$$\begin{aligned}
&(b \cdot n_1 - c \cdot m_1) \cdot (a \cdot \delta_{1,1} + b \cdot \delta_{1,2} + c \cdot \delta_{1,3}) \cdot v \cdot w + \\
&(c \cdot l_2 - a \cdot n_2) \cdot (a \cdot \delta_{2,1} + b \cdot \delta_{2,2} + c \cdot \delta_{2,3}) \cdot w \cdot u + \\
&(a \cdot m_3 - b \cdot l_3) \cdot (a \cdot \delta_{3,1} + b \cdot \delta_{3,2} + c \cdot \delta_{3,3}) \cdot u \cdot v = 0 \qquad (1)
\end{aligned}$$

This equation corresponds to a circumconic of ABC. Note that this circumconic does not depend on the choice of Q. Therefore the locus of the perspectors P of ABC and all built triangles A"B"C", homothetic to A'B'C' and perspective to ABC, is a fixed circumconic $\mathfrak{C}$ of ABC.

Expressions for coordinates of the center and perspector of this conic are simplified if coefficients of $v \cdot w, w \cdot u, u \cdot v$ in (1) are replaced with F_1, F_2, F_3, respectively, i.e.,

$$\begin{aligned}
F_1 &= (b \cdot n_1 - c \cdot m_1) \cdot (a \cdot \delta_{1,1} + b \cdot \delta_{1,2} + c \cdot \delta_{1,3}) \\
F_2 &= (c \cdot l_2 - a \cdot n_2) \cdot (a \cdot \delta_{2,1} + b \cdot \delta_{2,2} + c \cdot \delta_{2,3}) \\
F_3 &= (a \cdot m_3 - b \cdot l_3) \cdot (a \cdot \delta_{3,1} + b \cdot \delta_{3,2} + c \cdot \delta_{3,3})
\end{aligned}$$

With this notation the perspector K of the conic is

$$K = F_1 : F_2 : F_3$$

and its center O is:

$$O = F_1 \cdot (-a \cdot F_1 + b \cdot F_2 + c \cdot F_3) : F_2 \cdot (a \cdot F_1 - b \cdot F_2 + c \cdot F_3) : F_3 \cdot (a \cdot F_1 + b \cdot F_2 - c \cdot F_3)$$

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