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CIRCUMCEVIAN INVERSION PERSPECTOR

A Generalisation of the Triangle Center X(35)

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ABSTRACT. The triangle center $X(35)$ as mentioned in the Kimberling's Encyclopedia of Triangle Centers, is the point of concurrence of the lines joining the vertices of the triangle to the inverses of the opposite excentres in the circumcircle. In this paper, we obtain a generalization of the same.

NOTATION

Let ABC be a triangle. We denote its side lengths BC, CA and AB by a, b, c, its perimeter by $2s$, s being the semiperimeter, and its area by Δ. Its well-known triangle centers are the circumcenter O, the incenter I, the orthocenter H and the centroid G. The nine-point center (N) is the center of the nine-point circle and is the midpoint of segment OH.

I_a represents the excenter opposite to A and define I_b and I_c cyclically. The inradius is represented by r and the circumradius by R. The Euler Line is the line on which the circumcenter, the centroid and the orthocenter lie.

Na represents the Nagel Point, i.e. the point of concurrence of the lines joining the vertices of the triangle to the points of tangency of the corresponding excircles to the opposite sides.

(ABC) represents the circumcircle of triangle ABC. $Pow_{\omega}(X)$ represents the power of the point X wrt the circle ω.

The ordered triple $x : y : z$ represents the point with barycentric coordinates

$$\left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right)$$

$f(a, b, c) : f(b, c, a) : f(c, a, b)$ can be represented as $f(a, b, c)$ ::.

The point $X(k)$ represents the triangle center $X(k)$ as listed in the Kimberling's Encyclopedia of Triangle Centers.

CONWAY'S NOTATION

$S = 2\Delta$ and $S_A = {(b^2 + c^2 - a^2)}/{2} = bc\cos A$.

S_B and S_C are defined cyclically.

$S_{\omega} = S\cot\omega$, $\cot(\omega) = {(a^2 + b^2 + c^2)}/{2S}$, where ω is the Brocard angle.

Throughout this paper, the above notation shall be used. The symbols have the same meaning as in this section unless specified otherwise.

THE TRIANGLE CENTER $X(35)$

The triangle center $X(35)$ as mentioned in the Kimberling's Encyclopedia of Triangle Centers, is the point of concurrence of the lines joining the vertices of the triangle to the inverses of the opposite excentres in the circumcircle.

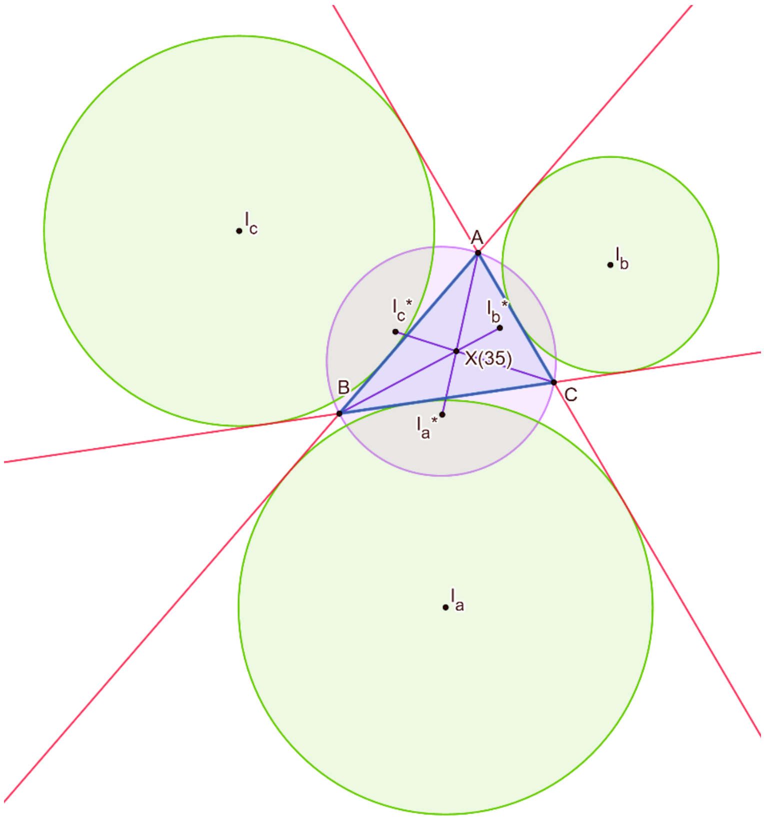

Figure 1. Triangle Center X(35)

The Incenter-Excenter Lemma

We begin with the well-known Incenter-Excenter Lemma that relates the incenter and excenters of a triangle.

***Lemma**. Let ABC be a triangle. Denote by L the mid-point of arc BC not containing A. Then, L is the center of the circle passing through I, I_a, B, C.*

Proof. Note that the points A, I, I_a and L are collinear (as L lies on the angle bisector of $\angle BAC$). We wish to show that $LB = LI = LC = LI_a$.

First, notice that $\angle LBI = \angle LBC + \angle CBI = \angle LAC + \angle CBI = \angle IAC + \angle CBI = {}^1/_2\,\angle A + {}^1/_2\,\angle B$.

However, $\angle BIL = \angle BAI + \angle ABI = {}^1/_2\,\angle A + {}^1/_2\,\angle B = \angle LBI$. Thus, triangle BIL is isosceles with $LB = LI$. Similarly, $LI = LC$. Hence, L is the circumcenter of triangle IBC. The circle passing through I, B and C must pass through I_a as well (the quadrilateral ICI_aB is cyclic as the opposite angles add up to $180°$). Thus, L is the center of the circle passing through I, I_a, B, C.

Using this property that relates the excentres and the incenter, we try to obtain a generalization of the triangle center $X(35)$.

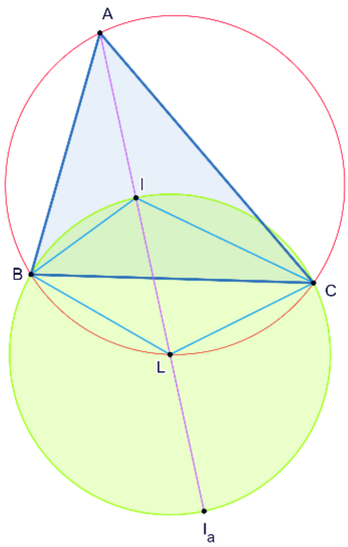

Figure 2. The Incenter-Excentre Lemma

The Circumcevian-inversion perspector

The following theorem is the crux of this paper.

Theorem (Circumcevian-inversion perspector). *The triangle formed by the inverses of reflections of a point in the vertices of its circumcevian triangle is perspective to the original triangle at a point called the Circumcevian-inversion perspector of that point with respect to the original triangle.*

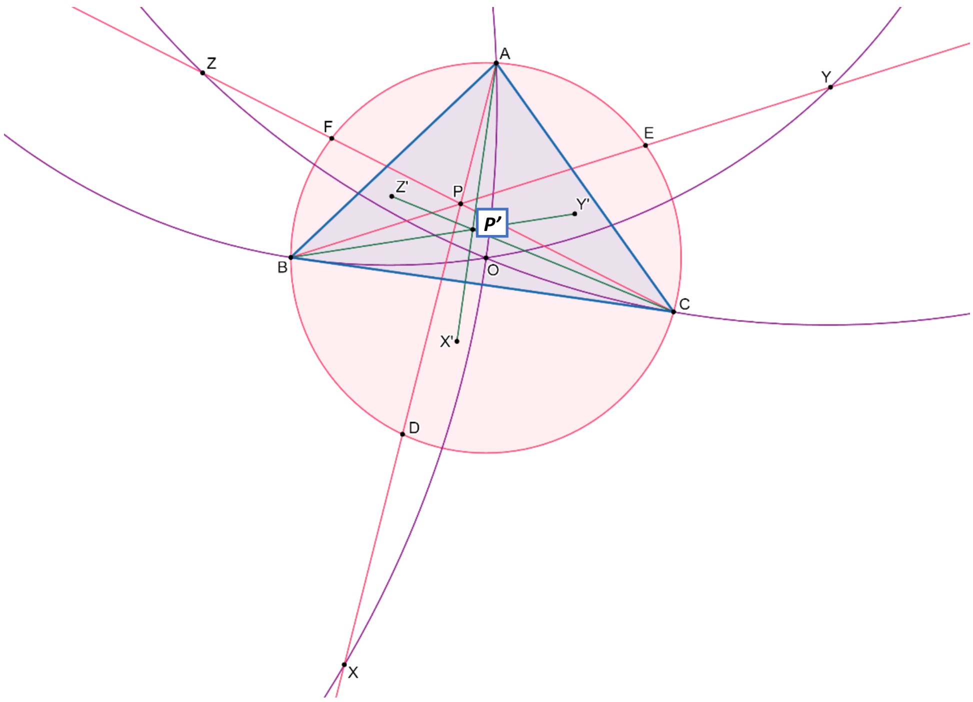

Figure 3. The Circumcevian-inversion perspector P'

In simple words, let P be any point in the plane of a triangle ABC and the lines AP, BP and CP intersect (ABC) again in D, E and F respectively. The reflection of P in D, E and F are X, Y and Z. The inverses of X, Y and Z in (ABC) are the points X', Y' and Z' respectively. Then the lines AX', BY' and CZ' are concurrent at a point which we shall call throughout this paper, the 'Circumcevian-inversion perspector' of P. The properties of the Circumcevian-inversion perspector of a point are discussed in the later sections.

Proof. Construct the circles $(AOX), (BOY), (COZ)$.

$$Pow_{(AOX)}(P) = AP \cdot PX = AP \cdot 2PD = 2Pow_{(ABC)}(P)$$

Similarly, we can prove that $Pow_{(BOY)}(P) = Pow_{(COZ)}(P) = 2Pow_{(ABC)}(P)$. So, the power of P and O wrt the three circles is equal. This implies that the three circles, $(AOX), (BOY), (COZ)$ are coaxial and their radical axis is the line OP. So, these circles share a second common point (other than O), say K.

Define I as the inversion about the circle (ABC). The image of any point X under this inversion is $I(X)$. Under the inversion, (AOX) is mapped to the line AX', (BOY) is mapped to the line BY', (COZ) is mapped to the line CZ'. The lines AX', BY', and CZ' share the points $I(O)$ and $I(K)$. Hence the lines AX', BY', and CZ' are concurrent at the point $I(K)$. (The other common point i.e. $I(O)$ is a point at infinity).

Remark. X, Y and Z can also be understood as the images of the point P in the homothety with factor -1 centered at the points D, E and F. If the factor of homothety is changed to some other number, in that case as well, the lines AX', BY' and CZ' are concurrent. The above proof with minor modifications works. If the factor of homothety is changed to h then, this perspector is called the h-circumcevian-inversion perspector of P with respect to the original triangle. The (-1)-circumcevian-inversion perspector is simply referred to as the circumcevian-inversion perspector.

It is clear by the incenter-excenter lemma that the triangle center $X(35)$ is the Circumcevian-inversion perspector of the incenter. Also, the circumcenter is the Circumcevian-inversion perspector of itself.

PROPERTIES OF THE CIRCUMCEVIAN-INVERSION PERSPECTOR OF A POINT

The Circumcevian-inversion perspector of a point displays interesting properties, three of which are listed below. The Circumcevian-inversion perspector opens the windows for many more triangle centers to be discovered.

Property 1. The Circumcevian-inversion perspector of the point P wrt triangle ABC lies on the line PO, O being the circumcenter of ABC.

Proof. Since the point K lies on the line PO, $I(K)$ must lie on PO as well.

Property 2. If the distance $PO = d$, and P' is the Circumcevian-inversion perspector of P, then ${PP'}/{P'O} = 1 - {d^2}/{R^2}$.

Proof.

$$Pow_{(ABC)}(P) = R^2 - d^2 = AP \cdot PD$$

$$Pow_{(AOX)}(P) = AP \cdot PX = 2AP \cdot PD = 2(R^2 - d^2) = OP \cdot PK$$

$$PK = \frac{2(R^2 - d^2)}{d} \Longrightarrow OK = \frac{2R^2 - d^2}{d}$$

$$OK \cdot OP' = R^2 \Longrightarrow OP' = \frac{R^2 d}{2R^2 - d^2} \Longrightarrow \frac{OP}{OP'} = \frac{2R^2 - d^2}{R^2} \Longrightarrow \frac{PP'}{P'O} = 1 - \frac{d^2}{R^2}$$

Property 3. P' and $I(P) = Q$ are harmonic conjugates wrt the points O and P.

Proof.

$$OP \cdot OQ = R^2 \Longrightarrow OQ = \frac{R^2}{d} \Longrightarrow PQ = \frac{R^2 - d^2}{d} \Longrightarrow \frac{PQ}{OQ} = \frac{R^2 - d^2}{R^2} = 1 - \frac{d^2}{R^2} \Longrightarrow \frac{PQ}{OQ} = \frac{PP'}{OP'}$$

Remark. The converse of property 3 is also easy to prove.

BARYCENTRIC COORDINATES OF THE CIRCUMCEVIAN-INVERSION PERSPECTOR OF A TRIANGLE CENTER

In this section, we find the barycentric coordinates of the Circumcevian-inversion perspector of a point by using Property 2 discussed in the last section.

Consider the point $P = (p, q, r)$. Note that here, we are considering the normalized barycentric coordinates of the point. The result can be denormalized later.

The circumcenter O has normalized barycentric coordinates $(^{a^2 S_A}/_{2S^2}, \_, \_)$ and the coordinates of P are (p, q, r). We use the barycentric distance formula to find the distance $d = PO$. Now, we use property 3.

$$d^2 = PO^2 = -\sum_{cyc} a^2 \left(\frac{b^2 S_b}{2S^2} - q\right)\left(\frac{c^2 S_c}{2S^2} - r\right)$$

$$\frac{PP'}{P'O} = 1 - \frac{d^2}{R^2} = 1 + \frac{\sum_{cyc} a^2 \left(\frac{b^2 S_b}{2S^2} - q\right)\left(\frac{c^2 S_c}{2S^2} - r\right)}{R^2}$$

$$P' = 8S^6 R^2 p + (4S^4 R^2 + \sum_{cyc} \left(a^2 (b^2 S_B - 2qS^2)(c^2 S_C - 2rS^2)\right)) a^2 S_A ::$$

After algebraic manipulations and denormalization, we get

$$P' = p(p + q + r)a^2 b^2 c^2 + a^2 (b^2 + c^2 - a^2)(a^2 qr + b^2 pr + c^2 pq) ::$$

It should be noted that the above expression is valid for all points $p: q: r$ instead of only for (p, q, r) as we have denormalised the expression.

SHINAGAWA COEFFICIENTS OF THE CIRCUMCEVIAN-INVERSION PERSPECTOR OF A TRIANGLE CENTER ON THE EULER LINE

Suppose that X is a triangle center on the Euler line given by barycentric coordinates $f(a, b, c) : f(b, c, a) : f(c, a, b)$. The Shinagawa coefficients of X are the functions $G(a, b, c)$ and $H(a, b, c)$ such that $f(a, b, c) = G(a, b, c) \cdot S^2 + H(a, b, c) \cdot S_B S_C$.

Many a times, Shinagawa coefficients of a triangle center can be conveniently expressed in terms of the following,

$$E = {(S_B + S_C)(S_C + S_A)(S_A + S_B)}/{S^2} = \left({abc}/{S}\right)^2 = 4R^2$$
$$F = {S_A S_B S_C}/{S^2} = {(a^2 + b^2 + c^2)}/{2} - 4R^2 = S_\omega - 4R^2$$

Let the Shinagawa coefficients of a triangle center P on the Euler line be (k, l). By property 1, the Circumcevian-inversion perspector of P i.e. P' lies on the Euler line as well. We try to find the Shinagawa coefficients of P'. O and H are circumcenter and orthocenter respectively of the reference triangle.

P has Shinagawa coefficients (k, l). Thus, $P = kS^2 + lS_B S_C$ ::.

If ${OP}/{PH} = {r_1}/{r_2}$, then $P = r_2 S^2 + (2r_2 - r_1)S_B S_C$ ::. Comparing the barycentric coordinates for P in the two cases, we get ${OP}/{PH} = {k + l}/{2k}$.

$$OP = \frac{(k + l)(OH)}{3k + l}$$

Now, we use property 3 of the Circumcevian-inversion perspector of a point.

$$\frac{PP'}{P'O} = 1 - \frac{d^2}{R^2} = 1 - \frac{(k + l)^2 (OH)^2}{(3k + l)^2 (R)^2} = 1 - \frac{(k + l)^2 (E - 8F)}{(3k + l)^2 E} = \frac{(3k + l)^2 E - (k + l)^2 (E - 8F)}{(3k + l)^2 E}$$

The barycentric section formula gives the Shinagawa coefficients (after cancelling out any common factors) of P' as

$$\left(2k(3k + l)E + (3k + l)^2 E - (k + l)^2 (E - 8F), 2l(3k + l)E - (3k + l)^2 E + (k + l)^2 (E - 8F)\right)$$

This formula can be used to find the Shinagawa coefficients of the Circumcevian-inversion perspector of any triangle center lying on the Euler line. Note that this formula is applicable because k and l have the same degree of homogeneity in a, b, c for any triangle center on the Euler line.

We find the Shinagawa coefficients of the Circumcevian-inversion perspector of the Nine Point Center and the De Longchamps Point.

Example 1. Nine Point Center i.e. $X(5)$ has the Shinagawa coefficients $(1,1)$. So, its Circumcevian-inversion perspector has the shinagawa coefficients $(5E + 8F, -E - 8F)$.

This is now $X(34864)$Kimberling's Encyclopedia of Triangle Centers.

Example 2. De Longchamps Point i.e. $X(20)$ has the Shinagawa coefficients $(1, -2)$. So, its Circumcevian-inversion perspector has the shinagawa coefficients $(E + 4F, -2E - 4F)$. Thus, the Circumcevian-inversion perspector of the triangle center $X(20)$ i.e. the De Longchamps Point is the triangle center $X(7488)$.

OTHER CIRCUMCEVIAN-INVERSION PERSPECTORS

CIRCUMCEVIAN-INVERSION PERSPECTOR OF THE CENTROID ($X(2)$)

According to the Kimberling's Encyclopedia of Triangle Centers,

$X(7496)$= $X(2), X(3)$- harmonic conjugate of $X(23)$

Barycentrics (Pending)

$X(23)$ is the inverse-in-circumcircle of $X(2)$.

By property 3, we can claim that $X(7496)$ is the Circumcevian-inversion perspector of $X(2)$ i.e. the centroid.

Barycentric formula gives the barycentric coordinates of G' as $(3a^2b^2c^2 + a^2(a^2 + b^2 + c^2)(b^2 + c^2 - a^2) ::)$.

Circumcevian-inversion perspector of the Orthocenter $(X(4))$

We claim that the triangle center $X(3520)$ is the Circumcevian-inversion perspector of the orthocenter i.e. $X(4)$. $X(186)$ is the inverse-in-circumcircle of the orthocenter (H). If we denote the Circumcevian-inversion perspector of H by H', then $X(4), X(3); H', X(186)$ is a harmonic bundle by property 3. $X(4) = (\tan A ::)$ and $X(3) = (\sin 2A ::)$. $X(186) = (\sin A\,(4\cos A - \sec A) ::) = (2\sin 2A - \tan A ::)$. By basic properties of barycentric coordinates in a harmonic bundle, we get $H' = (2\sin 2A + \tan A ::) = (\sin A\,(4\cos A + \sec A) ::) = X(3520)$.

Circumcevian-inversion perspector of the Symmedian Point $(X(6))$

According to the Kimberling's Encyclopedia of Triangle Centers,

$X(6)$ = inverse-in-circumcircle of $X(187)$

$X(574)$ = inverse-in-Brocard-circle of $X(187)$

Brocard circle of a triangle is the circle with the circumcenter and symmedian point of the triangle as its diametric ends.

This shows that $X(6), X(3); X(574), X(187)$ is in fact a harmonic bundle. By property 3, $X(574)$ is the Circumcevian-inversion perspector of the symmedian point.

Circumcevian-inversion perspector of the Nagel Point $(X(8))$

We take a different approach here, rather than just using the barycentric formula for circumcevian-inversion perspector.

It is known that the incenter is the complement of the Nagel point. Also, the nine-point center (N) of the triangle is the circumcenter of the medial triangle. By Feuerbach's Theorem, we have, $NI = {}^{R}/_{2} - r$.

Due to homothety of the triangle with its medial triangle, it follows that $ONa = 2 \cdot NI = R - 2r$.

Now, it is only a matter of using property 2 to find the barycentric coordinates Na' i.e. the Circumcevian-inversion perspector of Na.

$$\frac{NaNa'}{Na'O} = 1 - \frac{d^2}{R^2} = \frac{4Rr - 4r^2}{R^2}$$

$$O = X(3) = (\sin 2A ::)$$

$$Na = X(8) = (b + c - a ::)$$

$$Na' = (b + c - a + 4 \cdot \sin 2A\,(R - r) ::)$$

This is now $X(34866)$ in Kimberling's Encyclopedia of Triangle Centers.

Known Circumcevian-inversion perspectors

Triangle Center	**Circumcevian-inversion perspector**
Incenter	$X(35)$

Centroid	$X(7496)$
Circumcenter	Circumcenter
Orthocenter	$X(3520)$
Symmedian Point	$X(574)$
De Longchamps Point	$X(7488)$

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