Slow Beatty Sequences, Devious Convergence, and Partitional Divergence

Clark Kimberling and Kenneth B. Stolarsky

Abstract. Sequences $(\lfloor nr \rfloor)$ for $0 < r < 1$ are introduced as *slow Beatty sequences.* They and ordinary Beatty sequences (for which $r > 1$) provide examples of sequences that converge deviously (which at first might seem to diverge), as well as partitionally divergent sequences (which consist of convergent subsequences).

1. INTRODUCTION. Eleven years after receiving the first Ph.D. in mathematics ever granted by a Canadian university—and 19 years before presiding over the Canadian Mathematical Society and 8 more before donning the robe of the Chancellor of the University of Toronto—Samuel Beatty sent a problem proposal [2] to this MONTHLY. The proposal is often cast like this: if r and s are positive irrationals satisfying $1/r + 1/s = 1$, then the sequences $(\lfloor nr \rfloor)$ and $(\lfloor ns \rfloor)$ partition the positive integers. This is now known as Beatty's theorem, and the sequences as Beatty sequences (although Lord Rayleigh had published the theorem in 1894). Choosing r to be the golden ratio ($\varphi = (1 + \sqrt{5})/2$) gives the lower and upper Wythoff sequences, represented here as indexed in the Online Encyclopedia of Integer Sequences [4]:

$$\text{A000201} = (1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, \ldots)$$

$$\text{A001950} = (2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, 31, 34, 36, 39, \ldots).$$

Now suppose that r is an irrational number. If $r > 1$, the sequence $(\lfloor nr \rfloor)$ is a Beatty sequence, but not otherwise. If $0 < r < 1$, we shall call $(\lfloor nr \rfloor)$ a *slow Beatty sequence.* A close connection between the two is given by Theorem 1.

Theorem 1. *Suppose that t in $(0, 1)$ is irrational, and let $s(n) = \lfloor nt \rfloor$. Let a be the sequence of numbers n such that $s(n+1) = s(n)$, and b the sequence of those n such that $s(n+1) = s(n) + 1$. Then b is the Beatty sequence of $1/t$, and a is the Beatty sequence of $1/(1-t)$. Conversely, if c and d are a pair of Beatty sequences, say, of $1/t$ and $1/(1-t)$, and $s(n)$, a, and b are as before, then one of the following holds: $c = a$ and $d = b$, or $c = b$ and $d = a$.*

Proof. The condition $s(n+1) = s(n) + 1$ is equivalent to the presence of an integer between nt and $(n+1)t$. Suppose that $n = \lfloor m/t \rfloor$ for some m, and let $\varepsilon = m/t - n$. Then $0 < \varepsilon < 1$, so that

$$m < (m/t - \varepsilon)t + t = nt + t.$$

Also, $nt < m$, so that an integer between nt and $(n+1)t$ is m. There can be at most one such integer since $0 < t < 1$. Thus, $(\lfloor m/t \rfloor)$, for $m = 1, 2, \ldots,$ is the sequence b. By Beatty's theorem, a is the sequence $(\lfloor m/(1-t) \rfloor)$. For the converse, if $\lfloor nr \rfloor$

http://dx.doi.org/10.4169/amer.math.monthly.123.3.267
MSC: Primary 11B99

and $\lfloor ns \rfloor$ are arbitrary Beatty sequences, let $t = 1/r$, so that the two sequences are $\lfloor n/t \rfloor$ and $\lfloor n/(1-t) \rfloor$, both of which satisfy the description of $s(t)$ in the statement of the theorem. ■

Theorem 1 shows that the two kinds of Beatty sequences are so closely related that one should ask if slow Beatty sequences occur on their own. A wonderful example [1] is the definition of the rabbit constant (A014565):

$$\sum_{n=1}^{\infty} \frac{1}{2^{\lfloor n/\varphi \rfloor}} = [2, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots],$$

where the continued fraction consists of the numbers 2^{F_n}, where F_n is the nth Fibonacci number, for $n \geq -1$.

Another example springs from the infinite Fibonacci word

$$\text{A005614} = (1, 0, 1, 1, 0, 1, 0, 1, 1, 0, \ldots),$$

defined as the fixed point of the morphism $1 \to 10$, $0 \to 1$, starting with 1. The partial sums of the terms are $1, 1, 2, 3, 3, 4, 4, 5, 6, 6, \ldots$, given by the slow Beatty sequence $(\lfloor n/\varphi \rfloor)$ used to define the rabbit constant. The sequence is also closely related to the Hofstadter G-sequence, A005206.

Frequently, Beatty sequences occur in connection with inequalities; an example involving fractional parts follows:

$$\{n\varphi\} > 1/\varphi^2,$$

which holds if and only if $n = \lfloor k\varphi \rfloor$ for some positive integer k or $n = \lfloor k\varphi^2 \rfloor$ for some negative integer k. Another example comes from the inequality

$$2k^2 < n^2,$$

where k and n are integers. Here, for each n, the greatest such k is given by the slow Beatty sequence $(\lfloor n/\sqrt{2} \rfloor)$. This last inequality suggests many other slow Beatty sequences.

2. DEVIOUS CONVERGENCE. We turn now to quite a different feature of slow Beatty sequences. Let $t > 1$ and

$$g(n) = g(n, t) = \frac{n}{t \lfloor n/t \rfloor)}.$$

Taking $t = \varphi$, a numerical analyst might notice that $g(n) = F_{35}/(\varphi F_{34})$ for more than 1000 values of n. If nearly asleep, she might conclude that $(g(n))$ converges to $F_{35}/(\varphi F_{34})$. However, let L_n denote the nth Lucas number. Further checking shows that $g(n) = L_{58}/(\varphi L_{57})$ for more than a million values of n, even though, clearly, $(g(n))$ converges to 1. The slow and distinctive manner of convergence suggests a definition.

Definition. A sequence (x_n) *converges deviously* to L if (1) it converges to L, and (2) for every real B, there exists $\ell \neq L$ such that $x_n = \ell$ for more than B numbers n.

We next summarize without proof some well-known relevant facts for easy reference.

Lemma 1. *Let L_n and F_n denote the nth Lucas and nth Fibonacci numbers. Then*

1. $L_n = \varphi^n + (-\varphi)^n$,
2. $F_n = (\varphi^n - (-\varphi)^n)/\sqrt{5}$,
3. *if N and m are integers and $0 \le t < 1/N$, then $\lfloor N(m+t) \rfloor = Nm = N\lfloor m+t \rfloor$,*
4. *if x is real and m is an integer, then $\lfloor x - m \rfloor = \lfloor x \rfloor - m$.*

Theorem 2. *The sequence $(g(n)) = (g(n, \varphi))$ converges deviously to 1. Indeed,*

$$g(F_{2h+1}) = g(kF_{2h+1}) \text{ for } k = 1, 2, \ldots, L_{2h+1},$$
$$g(L_{2h}) = g(kL_{2h}) \text{ for } k = 1, 2, \ldots, F_{2h}.$$

Proof. We use the four items of Lemma 1 in the order written. First, $L_{2h+1} = \lfloor \varphi^{2h+1} \rfloor$ implies $1/\varphi^{2h+1} < 1/k$ for $k = 1, 2, \ldots, L_{2h+1}$. For these k,

$$F_{2h+1}\varphi = F_{2h+2} + 1/\varphi^{2h+1}$$

so that

$$\{F_{2h+1}\varphi\} < 1/k$$

and

$$\lfloor kF_{2h+1}\varphi \rfloor = k\lfloor F_{2h+1}\varphi \rfloor.$$

It follows that

$$\lfloor kF_{2h+1}(\varphi - 1) \rfloor = k\lfloor F_{2h+1}(\varphi - 1) \rfloor$$

and hence

$$\lfloor kF_{2h+1}/\varphi \rfloor = k\lfloor F_{2h+1}/\varphi \rfloor.$$

Therefore,

$$\frac{F_{2h+1}}{\varphi\lfloor F_{2h+1}/\varphi \rfloor)} = \frac{kF_{2h+1}}{\varphi\lfloor kF_{2h+1}/\varphi \rfloor)},$$

as desired.

To prove the second identity, we start with

$$\varphi < \frac{L_{2h+1}}{L_{2h}} + \frac{1}{F_{4h}},$$

for which the reader may supply a proof. Since $F_{4h} = L_{2h}F_{2h}$, we have $\varphi L_{2h} - L_{2h+1} < 1/F_{2h}$, so that $\{\varphi L_{2h}\} < 1/k$ for $k = 1, 2, \ldots, F_{2h}$. For these k,

$$\lfloor kL_{2h}/\varphi \rfloor = k\lfloor L_{2h}/\varphi \rfloor,$$

and the desired result follows. ■

It is natural to ask what happens when floor is replaced by ceiling. For $t > 0$, define

$$\widehat{g}(n) = \widehat{g}(n,t) = \frac{n}{t\lceil n/t\rceil}.$$

Theorem 3. *The sequence* $(\widehat{g}(n)) = (\widehat{g}(n), \varphi)$ *converges deviously to* 1. *Indeed,*

$$\widehat{g}(F_{2h}) = \widehat{g}(kF_{2h}) \textit{ for } k = 1, 2, \ldots, L_{2h} - 1,$$
$$\widehat{g}(L_{2h+1}) = \widehat{g}(kL_{2h+1}) \textit{ for } k = 1, 2, \ldots, F_{2h+1} - 1.$$

A proof similar to that of Theorem 2 can be developed from the identities $L_{2h} = \lceil \varphi^{2h+1} \rceil$ and

$$\varphi < \frac{L_{2h}}{L_{2h+1}} + \frac{1}{F_{4h}} + 1.$$

If $(g(n))$ is any sequence that converges deviously, then one still has, of course,

$$g(n+1) - g(n) \to 0.$$

However, if $k(n)$ is any function such that

$$h(n) := k(n)(g(n+1) - g(n) \nrightarrow 0,$$

then the sequence $(h(n))$ diverges (since it does tend to 0 on a subsequence). The simplest possibility in this case is that $(h(n))$ has two limit points. These observations are especially relevant for the particular $g(n)$ introduced above, as we shall see in the next section.

3. PARTITIONAL DIVERGENCE.

Definition. Suppose that w_1 and w_2 are any two sequences that partition the positive integers, and let $w = \{w_1, w_2\}$. Suppose further that (a_n) is a sequence such that $a_{w_1(n)} \to L_1$ and $a_{w_2(n)} \to L_2$, so that (a_n) converges if and only if $L_1 = L_2$. If $L_1 \neq L_2$, we shall say that (a_n) *diverges partitionally on* w.

We now take $k(n) = n$ and show that for our specific $g(n)$ the sequence given by

$$h(n) = n(g(n+1) - g(n))$$

diverges partitionally. Moreover, the relevant partition is a partition into Beatty sequences! (Also, we can take for t any irrational number greater than 1.)

Lemma 2. *Suppose that* $t > 1$ *is irrational and* n *is a positive integer. Then*

$$\left\lfloor \frac{\lfloor nt \rfloor}{t} \right\rfloor = n - 1 \quad \textit{and} \quad \left\lfloor \frac{\lfloor nt \rfloor + 1}{t} \right\rfloor = n, \tag{1}$$

and if $\widehat{t} = t/(t-1)$, *then*

$$\left\lfloor \frac{\lfloor n\widehat{t} \rfloor}{t} \right\rfloor = \left\lfloor \frac{n}{t-1} \right\rfloor \quad \textit{and} \quad \left\lfloor \frac{\lfloor n\widehat{t} \rfloor + 1}{t} \right\rfloor = \left\lfloor \frac{n}{t-1} \right\rfloor. \tag{2}$$

Proof. Of the four identities we prove only the third, as the other three have similar proofs. For $s > 0$ we have

$$\frac{\lfloor ns \rfloor}{s} \le n,$$

so

$$\lfloor ns \rfloor (1 + (1/s)) \le \lfloor ns + n \rfloor .$$

Set $s = 1/(t-1)$ to obtain

$$t \left\lfloor \frac{n}{t-1} \right\rfloor \le \left\lfloor \frac{nt}{t-1} \right\rfloor .$$

Divide by t and apply the floor function to obtain

$$\left\lfloor \frac{n}{t-1} \right\rfloor \le \left\lfloor \frac{\lfloor \widehat{nt} \rfloor}{t} \right\rfloor .$$

Since clearly also

$$\left\lfloor \frac{n}{t-1} \right\rfloor \ge \left\lfloor \frac{\lfloor \widehat{nt} \rfloor}{t} \right\rfloor ,$$

the third identity is proved. ∎

Theorem 4. *Suppose that $t > 1$ is irrational, so that the Beatty sequences given by $w_1(n) = \lfloor nt \rfloor$ and $w_2(n) = \lfloor nt/(t-1) \rfloor$ partition the positive integers. Then the sequence $(h(n))$ is partitionally divergent; indeed, $h(w_1(n)) \to 1 - t$ and $h(w_2(n)) \to 1$.*

Proof. Substitute $\lfloor nt \rfloor$ for n and apply (1):

$$\begin{aligned} h(w_1(n)) &= \frac{\lfloor nt \rfloor (\lfloor nt + 1 \rfloor)}{tn} - \frac{\lfloor nt \rfloor^2}{t(n-1)} \\ &= \frac{\lfloor nt \rfloor}{t} \left(\frac{nt - \varepsilon_n + 1}{n} - \frac{nt - \varepsilon_n}{n-1} \right), \end{aligned}$$

where $0 < \varepsilon_n < 1$. The last equation readily reduces to

$$h(w_1(n)) = \frac{\lfloor nt \rfloor}{nt} \left(\frac{n(1-t)}{n-1} + \frac{\varepsilon_n - 1}{n-1} \right),$$

so that $h(w_1(n)) \to 1 - t$. Next, let

$$\widehat{\varepsilon_n} = \frac{nt}{t-1} - \left\lfloor \frac{nt}{t-1} \right\rfloor = \text{fractional part of } \widehat{nt}.$$

A calculation similar to that above but using (2) shows that

$$h(w_2(n)) = (\frac{nt}{t-1} - \widehat{\varepsilon}_n)\left(\frac{\left\lfloor \frac{nt}{t-1} \right\rfloor + 1}{t\left\lfloor \frac{n}{t-1} \right\rfloor} - \frac{\left\lfloor \frac{nt}{t-1} \right\rfloor}{t\left\lfloor \frac{n}{t-1} \right\rfloor}\right)$$

$$= (\frac{nt}{t-1} - \widehat{\varepsilon}_n)(\frac{1}{t\left\lfloor \frac{n}{t-1} \right\rfloor})(1)$$

$$= \frac{n}{(t-1)\left\lfloor \frac{n}{t-1} \right\rfloor} - \frac{\widehat{\varepsilon}_n}{t\left\lfloor \frac{n}{t-1} \right\rfloor},$$

so that $h(w_2(n)) \to 1$. ■

Of course the definition of partitional divergence extends to partitions into more than two classes. For a simple example involving three classes, consider the previously mentioned Beatty sequences A = A000201 and B = A001950. The complement of B is A, which partitions into the composite sequences $AA = (1, 4, 6, 9, 12, \ldots)$ and $AB = (3, 8, 11, 16, \ldots)$; viz., if $a(n) = \lfloor n\varphi \rfloor$ and $b(n) = \lfloor n\varphi^2 \rfloor$ are the nth terms of the lower and upper Wythoff sequences, respectively, then $a(a(n))$ is the nth term of AA and $a(b(n))$ is the nth term of AB. We shall create a sequence that diverges partitionally according to the three classes B, AA, AB. Let

$$f(n) = \lceil n/\varphi \rceil \,/\, \lfloor n/\varphi \rfloor\,;$$
$$s(n) = \varphi f(n)(f(n+2) - 2f(n+1) + f(n)).$$

Theorem 5. *The sequence* $(s(n))$ *diverges partitionally. Specifically,* $s(AA)$ *is the constant sequence* $(\varphi, \varphi, \ldots)$*;* $s(B)$ *is the constant sequence* $-s(AA)$*; and* $s(AB)(n) = 2\varphi/(1 + n + \lfloor n\varphi \rfloor)$*, which converges to* 0.

A proof of Theorem 5 is straightforward and omitted. It is interesting that AA is column 1 of the Wythoff array W (described at A035513), B is the ordered sequence of all the terms in the even numbered columns of W, and AB likewise matches the odd numbered columns, except the first.

Remark. For a guide to much of the literature on Beatty sequences, see the introductory remarks of [3] and the references [3, 12, 14] of [3].

REFERENCES

1. P. G. Anderson, T. C. Brown, P. S.-S. Shiue, A simple proof of a remarkable continued fraction identity, *Proc. Amer. Math. Soc.* **123** (1995) 2005–2009.
2. S. Beatty, Problem 3173, *Amer. Math. Monthly* **33** (1926) 159, http://www.jstor.org/stable/2300153.
3. K. O'Bryant, Sturmian words and the permutation that orders fractional parts, *J. Algebraic Combin.* **19** (2004) 91–115.
4. Online Encyclopedia of Integer Sequences: https://oeis.org/. (Searching for "Beatty sequence" finds hundreds of Beatty sequences.)

CLARK KIMBERLING received his Ph.D. from the Illinois Institute of Technology in 1970 under the supervision of Abe Sklar. Ever since, he has been on the mathematics faculty at the University of Evansville, where he manages the Encyclopedia of Triangle Centers (http://faculty.evansville.edu/ck6/encyclopedia/ETC.html) and contributes to the Online Encyclopedia of Integer Sequences (https://oeis.org/).
Department of Mathematics, University of Evansville, Evansville, IN 47722, USA
ck6@evansville.edu

KENNETH B. STOLARSKY received his undergraduate education at Caltech, and obtained a Ph.D. from the University of Wisconsin (Madison) in 1968 under the supervision of Marvin I. Knopp. He did one year of postdoctoral work at the Institute for Advanced Study (Princeton) and has been (aside from sabbaticals) at the University of Illinois (Urbana-Champaign) since then, where he is now professor emeritus of mathematics. In 2006 he and Karl Dilcher received a Lester R. Ford award for their 2005 paper "A Pascal-Type Triangle Characterizing Twin Primes." His research interests include number theory, combinatorics, inequalities, extremal problems of geometry, and complex analysis (with emphasis on polynomials).
Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
Stolarsk@math.uiuc.edu

100 Years Ago This Month in The American Mathematical Monthly
Edited by Vadim Ponomarenko

An interesting article on "University Registration and Statistics" appeared in *Science* on January 21. The registrations from thirty of the larger universities, including the large endowed universities and most of the state universities of the middle west, are compiled. These tables show a total registration in September, 1915, of 100,514 students, or approximately one student from each thousand of population in the United States. This student body is governed and instructed by more than 12,000 officers and instructors, or about one officer or instructor to every eight students. During the summer sessions of 1915, the thirty institutions report registrations of 35,652 students. For the year 1915–1916, the following are the eight universities with largest registrations: Columbia (7,042); Pennsylvania (6,655); California (5,977); New York University (5,853); Michigan (5,821); Illinois (5,511); Harvard (5,435); Cornell (5,392).

[The 30 biggest U.S. colleges in Fall 2013 together enrolled 2,708,792 students. -Eds.]

—Excerpted from "Notes and News" **23** (1916) 97–100.