RESIDUALLY SMALL COMMUTATIVE RINGS

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Abstract. Let R be a ring. Following the literature, R is called *residually finite* if for every $r \in R \backslash \{0\}$, there exists an ideal I_r of R such that $r \notin I_r$ and R/I_r is finite. In this note, we define a *strictly* infinite commutative ring R with identity to be *residually small* if for every $r \in R \backslash \{0\}$, there exists an ideal I_r of R such that $r \notin I_r$ and $|R/I_r| < |R|$. The purpose of this article is to study such rings, extending results on (infinite) residually finite rings.

1. Introduction

We begin by recalling that a group G is *residually finite* if for every $g \in G \backslash \{e\}$, there exists a normal subgroup H_g of G such that $g \notin H_g$ and G/H_g is finite. Residually finite groups are objects of great interest; for good overviews see Hartley [7], Magnus [14], and Segal [21]. In Lewin [13] and Orzech & Ribes [20], the concept of residual finiteness is stated for rings in the natural way. Specifically, a ring R is called *residually finite* if for every $r \in R \backslash \{0\}$, there exists an ideal I_r of R such that $r \notin I_r$ and R/I_r is finite. In these papers, the authors translate many of the previous results on residually finite groups to rings.

Years later, Varadarajan ([23]–[24]) extends the notion of residual finiteness to modules. He calls a left (right) module M over a ring R a *residually finite module* if for every nonzero $m \in M$, there is a submodule N_m of M not containing m such that M/N_m is finite. Moreover, he makes some connections between residual finiteness in groups and the more recent modification for rings above. For instance, in Baumslag [3], the author shows that if G is a finitely generated, residually finite group, then the group $Aut(G)$ of automorphisms of G is residually finite. Varadarajan later establishes the following ([23], Theorem 1.2): let R be a finitely generated residually finite ring. Then the group $Aut(R)$ of ring automorphisms of R is a residually finite group. He also defines a ring R to be an RRF ring (LRF ring) if every right R-module (left R-module) is residually finite. Somewhat confusingly (given the terminology introduced earlier by Lewin, Orzech, and Ribes), Varadarajan calls a ring an RF ring if it is simultaneously an RRF ring and an LRF ring. Faith simplified and generalized some of Varadarajan's results in [5], retaining Varadarajan's definitions of RRF ring and LRF ring.

The works cited above on residual finiteness of rings consider general associative rings. *In this paper, we restrict our focus to commutative rings with identity* $1 \neq 0$*. Unless otherwise specified, all rings are assumed commutative with nonzero identity.* Our purpose is to extend the notion

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of residual finiteness to other cardinalities. Of central importance in this note is the following definition:

Definition 1. *Let R be an* infinite *ring (that is, R has infinite cardinality). Then we say that R is residually small if for every nonzero $r \in R$, there exists an ideal I_r of R not containing r such that $|R/I_r| < |R|$.*

We pause to justify why we consider only infinite rings in the previous definition. Observe that every finite ring is residually finite: if R is a finite ring and $r \in R \backslash \{0\}$, then $r \notin \{0\}$ and $R/\{0\}$ is finite. However, not every finite ring is residually small (use Definition 1, but omit the infinitude assumption). Let R be a finite ring. Then R is not residually small if and only if there is a nonzero $r \in R$ such that for all ideals I_r of R not containing r, $|R/I_r| = |R|$. Since R is finite, this reduces to the assertion that there is a nonzero $r \in R$ such that the only ideal of R not containing r is the zero ideal. But then the nonzero ideal Rr is contained in every nonzero ideal of R. Since R is finite, R has but finitely many ideals. In any ring with but finitely many ideals, it is easy to show that R has a minimum ideal (that is, a nonzero ideal contained in every nonzero ideal) if and only if R is uniform. Thus a finite ring R is residually small if and only if R is not uniform.

Definition 1 generalizes the notion of residual finiteness for infinite rings in that all infinite residually finite rings are residually small. Definition 1 is also closely related to the concept of a homomorphically smaller module. In Oman & Salminen [17] the authors, borrowing terminology introduced in Tucci [22], define a (unitary) module M over a commutative ring R to be *homomorphically smaller* (HS for short) if and only if M is infinite and $|M/N| < |M|$ for every nonzero submodule N of M. We also define a ring R to be an *HS ring* if R is an HS module over itself, that is, if R is infinite and $|R/I| < |R|$ for every nonzero ideal I of R. The definition of an HS ring extends the notion of (infinite, commutative) residually finite rings appearing in Chew & Lawn [4][1]. In Levitz and Mott [12], the authors extend Chew and Lawn's results to rings without identity and say such rings have the *finite norm property*. This concept was also considered by Ion, Militaru, and Niţă in [8][2] and [9]. To mitigate confusion, throughout the remainder of the article, if we state that a ring R is residually finite, we always mean that for any nonzero $r \in R$, there exists an ideal I_r not containing r such that R/I_r is finite.

The outline of this article is as follows. In Section 2, we prove fundamental results on residually small rings. Section 3 is devoted to presenting the main results of this paper. Finally, Section 4 concludes the article with some open questions.

2. Preliminaries

To initiate the reader, we present some natural examples of HS rings.

[1] The definition of "residually finite ring" is, quite unfortunately, not unique in the literature. In [4], the authors define a (not necessarily commutative) ring R to be residually finite if R/I is finite for every nonzero two-sided ideal I of R.

[2] In this paper, the authors call an infinite, non-simple unital ring R a *finite quotient ring* if R/I is finite for every nonzero two-sided ideal I of R. Using Chew and Lawn's terminology, a finite quotient ring is an infinite, non-simple residually finite ring.

Example 1 ([17], Examples 2.1 and Lemma 2.2). *The following domains are HS rings:*

(1) *infinite fields,*
(2) *the ring $\mathbb{Z}$ of integers,*
(3) *the polynomial ring $k[X]$, where k is a finite field, and*
(4) *the power series ring $\mathbb{Q}[[T]]$.*

Sketch of Proof. We remark on each item above in succession.

(1) Obvious.

(2) Obvious.

(3) Let k be a finite field. As is well-known, $k[X]$ is a PID. Let $f(X) \in k[X]$ be a nonzero nonunit of degree n. Then $k[X]/\langle f(x)\rangle$ forms a vector space over k of dimension n. Since k is finite, $k[X]/\langle f(X)\rangle$ is too.

(4) If I is a proper nonzero ideal of $\mathbb{Q}[[T]]$, then $I = \langle T^n\rangle$ for some positive integer n. Thus modulo I, we obtain polynomials in T over $\mathbb{Q}$ of degree at most $n-1$. This shows that $\mathbb{Q}[[T]]/I$ is countable, whereas $|\mathbb{Q}[[T]]| = 2^{\aleph_0}$. Therefore, $\mathbb{Q}[[T]]$ is HS. □

Next, we recall two definitions that will facilitate our investigations. In Oman [16], the author defines an ideal I of a (not necessarily commutative) ring R to be *small* if $|I| < |R|$ and *large* if $|R/I| < |R|$. With this terminology, an infinite ring R is HS if and only if all nonzero ideals of R are large. We can now categorize the residually small rings in terms of large ideals as follows:

Proposition 1. *An infinite ring R is residually small if and only if the intersection of the large ideals of R is trivial.*

Proof. Let R be an infinite ring. Then R is not residually small if and only if there exists $r \in R\backslash\{0\}$ such that every large ideal of R contains r if and only if the intersection of the large ideals of R is nontrivial. □

Remark 1. *Let D be a domain, and let $\mathcal{L}(D)$ denote the collection of large ideals of D. If D is residually small, then the previous proposition implies that $M := \bigoplus_{I\in\mathcal{L}(D)} D/I$ is a faithful torsion D-module. Using the terminology introduced in Oman & Schwiebert* [18]–[19], *every residually small domain is an FT ring. Additionally, we refer the reader to Anderson & Chun* [1]–[2] *for further reading on torsion modules over commutative rings.*

We now establish that neither the class of HS rings nor the class of residually small rings contains the other. Moreover, we show that the class of residually small rings properly contains the class of infinite residually finite rings.

Proposition 2. *The following hold:*

(1) *there exist HS rings which are not residually small,*
(2) *there exist residually small rings which are not HS, and*
(3) *there exist residually small rings which are not residually finite.*

Proof. To establish (1), simply observe that an infinite field is HS but is not residually small.

As for (2), consider the ring $\mathbb{Z} \times \mathbb{Z}$, and let $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ be nonzero. We can assume without loss of generality that $m \neq 0$. Now pick a prime p such that $p \nmid m$. Then $(m, n) \notin p\mathbb{Z} \times \mathbb{Z}$ and

$(\mathbb{Z}\times\mathbb{Z})/(p\mathbb{Z}\times\mathbb{Z})\cong\mathbb{Z}/p\mathbb{Z}$, which is finite. Thus $\mathbb{Z}\times\mathbb{Z}$ is residually small. On the other hand, note that

$$|(\mathbb{Z}\times\mathbb{Z})/(\mathbb{Z}\times\{0\})| = |\mathbb{Z}| = |\mathbb{Z}\times\mathbb{Z}|.$$

Thus $\mathbb{Z}\times\mathbb{Z}$ is not HS.

Finally, we come to (3). Consider again the ring $\mathbb{Q}[[T]]$. Suppose $f(T) := \sum_{i=0}^{\infty} a_i T^i$ is a nonzero element of $\mathbb{Q}[[T]]$. Now let n be least such that $a_n \neq 0$. Then $f(T) \notin \langle T^{n+1}\rangle$ and (by (4) of Example 1) $\mathbb{Q}[[T]]/\langle T^{n+1}\rangle$ is countable. We deduce that $\mathbb{Q}[[T]]$ is residually small. Lastly, if I is any proper ideal of $\mathbb{Q}[[T]]$, then $\mathbb{Q}$ naturally embeds into $\mathbb{Q}[[T]]/I$. Thus $\mathbb{Q}[[T]]$ is not residually finite. □

In our first theorem we prove that, with the exception of fields, the class of residually small rings properly contains the class of HS rings.

Theorem 1. *Let R be an HS ring. Then R is residually small if and only if R is not a field.*

Proof. Suppose that R is an HS ring. As we have noted, if R is a field, then R is not residually small. Conversely, assume that R is not a field. By Proposition 3.2 of [17], $Ann(R) := \{r \in R\colon rR = \{0\}\}$ is a prime ideal of R. Since R has an identity, we see that $Ann(R) = \{0\}$. Therefore, R is a domain. Now, let $r \in R\backslash\{0\}$ be arbitrary, and suppose by way of contradiction that $|R/I| = |R|$ for every ideal I of R not containing r. Since R is HS, it follows that the only ideal of R not containing r is $\{0\}$. Thus the principal ideal Rr is contained in every nonzero ideal of R. As R is not a field, we deduce that r is not a unit. But because R is a domain, $Rr^2 \subsetneq Rr$, and we have a contradiction to the minimality of Rr. □

Before presenting residually small-themed results for general rings, we focus first on those which are Noetherian. For assistance, we shall require the following lemmas.

Lemma 1 (Oman [15], Lemma 3)**.** *Let R be a ring and I a finitely generated ideal of R. Then*

(1) *if R/I is finite, then R/I^n is finite for every positive integer n;*
(2) *if R/I has infinite cardinality κ, then R/I^n has cardinality κ for every positive integer n.*

Lemma 2 ([20], Lemma 3)**.** *Let R be a Noetherian ring and let x be a nonzero element of R. Then there is a maximal ideal J of R and a positive integer n such that $x \notin J^n$.*

It is easy to see that the ring $\mathbb{Z}$ of integers is residually finite. More generally, we prove

Proposition 3. *Let R be an infinite Noetherian ring such that $|R/J| < |R|$ for every maximal ideal J of R. Then R is residually small.*

Proof. Let R be as stated. If $x \in R$ is nonzero, then by Lemma 2, there exists a maximal ideal J and positive integer n such that $x \notin J^n$. Since $|R/J| < |R|$ and R is infinite, Lemma 1 implies that $|R/J^n| < |R|$. We conclude that R is residually small. □

Corollary 1. *Let R be an infinite Noetherian local ring with maximal ideal J. Then R is residually small if and only if $|R/J| < |R|$.*

Proof. If R is residually small, then there exists a proper ideal I of R such that $|R/I| < |R|$. Since $I \subseteq J$, also $|R/J| < |R|$ (note that there is a natural surjection of R/I onto R/J). The converse holds by Proposition 3. □

Next, we show that we can weaken the hypothesis of Proposition 3 if the Noetherian ring R is a domain.

Theorem 2. *Suppose that D is a Noetherian domain. If D has at least one proper large ideal, then D is residually small. Moreover, if D has a proper nonzero ideal I of finite index in D, then D is residually finite.*

Proof. Let D be a Noetherian domain and suppose that $I \neq D$ is a large ideal of D (hence D is not a field). By Krull's Intersection Theorem,

$$\bigcap_{n=1}^{\infty} I^n = \{0\}. \tag{2.1}$$

Since I is large and D is Noetherian (and infinite, since D is not a field), we apply Lemma 1 to conclude that I^n is large for every positive integer n. This fact along with (2.1) above implies that the intersection of all large ideals of D is trivial. Therefore, D is residually small by Proposition 1.

Now suppose that I is a proper nonzero ideal of D such that D/I is finite, and let $d \in D$ be nonzero. By (2.1), there is a positive integer n such that $d \notin I^n$. Applying (1) of Lemma 1 concludes the proof. □

Remark 2. *One cannot dispense with the assumption that D is a domain in the statement of the previous theorem. For example, consider the ring $R := \mathbb{F}_2 \times \mathbb{Q}$. Then R is Noetherian and $\{0\} \times \mathbb{Q}$ is the unique proper large ideal of R. Proposition 1 implies that R is not residually small.*

We now work toward classifying the infinite cardinals ρ for which there exists a residually small Noetherian domain of cardinality ρ. Again, we require a lemma.

Lemma 3 (Kearnes & Oman [11], Lemmas 2.1 and 2.2)**.** *Let ρ be an infinite cardinal and let $\kappa < \rho$ be a cardinal which is either a prime power or is infinite. Then there exists a Noetherian domain D of size ρ with a residue field of size κ if and only if $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$.*

Proposition 4. *Let ρ be an infinite cardinal. There exists a Noetherian residually small domain of size ρ if and only if there exists a cardinal κ satisfying $\kappa < \rho \leq \kappa^{\aleph_0}$.*

Proof. Let ρ be an infinite cardinal. Suppose first that D is a Noetherian residually small domain of size ρ. Then there exists a proper ideal I of D such that $|D/I| < |D|$. Further, $I \subseteq J$ for some maximal ideal J of D. Thus $|D/J| \leq |D/I| < |D|$. Now set $\kappa := |D/J|$. Then Lemma 3 implies that $\kappa < \rho \leq \kappa^{\aleph_0}$. Conversely, suppose that $\kappa < \rho \leq \kappa^{\aleph_0}$. If κ is infinite, then Lemma 3 furnishes us with a Noetherian domain D with a residue field of size κ. By Theorem 2, D is residually small. Otherwise, κ is finite. Pick a prime q with $\kappa \leq q$. Then we see that $q < q + \aleph_0 \leq \rho \leq q^{\aleph_0}$, and invoking Lemma 3 again, we obtain a Noetherian domain D of size ρ with a residue field of size q. A final application of Theorem 2 concludes the proof. □

We can actually strengthen the previous proposition somewhat by showing that for any infinite cardinal ρ satisfying $\kappa < \rho \leq \kappa^{\aleph_0}$ for some cardinal κ, there exists a Noetherian residually small domain of size ρ which is not HS. To do this, we require another lemma.

Lemma 4. *Suppose that D is a Noetherian residually small domain. Then the polynomial ring $D[X]$ is residually small but not HS. Furthermore, $D[X]$ is residually finite if and only if D is residually finite.*

Proof. Suppose D is residually small and Noetherian, and let I be a proper large ideal of D. Then we have a sequence $D[X] \to D \to D/I$ of surjective ring maps. Letting K be the kernel of the composition, we see that $|D[X]/K| = |D/I| < |D| = |D[X]|$. Hence K is a proper large ideal of $D[X]$. Since $D[X]$ is a Noetherian domain, we may apply Theorem 2 to conclude that $D[X]$ is residually small. Lastly, observe that $|D[X]/\langle X\rangle| = |D| = |D[X]|$; thus $D[X]$ is not HS.

Suppose now that $D[X]$ is residually finite. Then there is a proper ideal I of $D[X]$ such that $D[X]/I$ is finite. Moreover, there is a natural embedding of $D/(D \cap I)$ into $D[X]/I$. Observe that $D \cap I$ is nontrivial, lest D be a finite field (and therefore not residually small). Moreover, $D \cap I$ is a proper ideal of D, lest $I = D[X]$. Conversely, assume that D is residually finite, and let J be a proper ideal of D of finite index. As above, the canonical surjections $D[X] \to D \to D/J$ yield an ideal I of $D[X]$ for which $D[X]/I \cong D/J$. Thus I is proper and nonzero. Applying Theorem 2 completes the argument. □

Corollary 2. *Let ρ be an infinite cardinal. There exists a Noetherian residually small domain D of size ρ if and only if there exists a cardinal κ satisfying $\kappa < \rho \leq \kappa^{\aleph_0}$. In this case, we can find such a D which is not HS.*

A natural question arises. Is it the case that for any infinite cardinal ρ, there exists a residually small domain D of cardinality ρ which is not HS? We end this section with the answer.

Proposition 5. *Let ρ be an infinite cardinal. There exists a residually small domain D of size ρ which is not HS.*

Proof. If $\rho = \aleph_0$, then (applying Lemma 4) we may take $D := \mathbb{Z}[X]$; noting that $D/\langle X\rangle$ is infinite, D is not HS. Suppose now that $\rho > \aleph_0$, and let R be any domain of size less than ρ. Now set $D := R[X_i \colon i < \rho]$, the polynomial ring over R in ρ many variables, and let $f \in D$ be nonzero. Without loss of generality, we may assume that $f \in R[X_0, \ldots, X_n]$ for some $n < \omega < \rho$. Observe that $D = (R[X_0, \ldots, X_n])[X_i \colon n < i < \rho]$. Thus there is a natural ring surjection $\varphi \colon D \to R[X_0, \ldots, X_n]$. Setting $K := ker(\varphi)$, we have $D/K \cong R[X_0, \ldots, X_n]$. It follows that K is large. Now, φ is the identity map on $R[X_0, \ldots, X_n]$; since $f \in R[X_0, \ldots, X_n]$ is nonzero, $\varphi(f) = f \notin K$. This proves that D is residually small. As $|D/\langle X_0\rangle| = |D|$, we see that D is not HS. □

Corollary 3. *Every ring is a subring of a residually small ring; every domain is a subring of a residually small Noetherian domain.*

Proof. Let R be a ring, and let ρ be an uncountable cardinal larger than $|R|$. Then, as the proof of Proposition 5 shows, $R[X_i \colon i < \rho]$ is residually small (one need not assume that R is a domain). Suppose now that D is a domain. Choose an ordinal a such that $|D| \leq \aleph_{a+\omega}$. Then of course, $S :=$

$D[X_i \colon i < \aleph_{a+\omega}]$ is a domain which contains D and has cardinality $\aleph_{a+\omega}$. Let F be the quotient field of S. Then also $|F| = \aleph_{a+\omega}$. Moreover, the power series ring $F[[T]]$ is a DVR (hence Noetherian) and has cardinality $(\aleph_{a+\omega})^{\aleph_0} > \aleph_{a+\omega}$ (this inequality is immediate from König's Theorem, since $\aleph_{a+\omega}$ has countable cofinality). Now, $\aleph_{a+\omega} = |F| = |F[[T]]/\langle T\rangle| < |F[[T]]| = (\aleph_{a+\omega})^{\aleph_0}$. We invoke Theorem 2 to conclude the proof. □

3. Main results

The goal of this section is to investigate how the residually small property behaves relative to polynomial ring extensions, direct products, quotient rings, factor rings, and integral extensions. We commence our investigations with polynomial rings, showing that residual smallness passes nicely to polynomial extensions. The following theorem significantly extends Lemma 4 as well as Theorem 2.2 of [23], where the author proves (among other things) that a ring R is residually finite if and only if $R[X]$ is residually finite.

Theorem 3. *Let R be an infinite ring, and let ρ be a nonzero cardinal. Then the following hold.*

(1) *If R is residually small, then $R[X_i \colon i < \rho]$ is residually small.*
(2) *If $R[X_i \colon i < \rho]$ is residually small and $\rho \leq |R|$, then R is residually small.*
(3) *If $\rho > |R|$, then $R[X_i \colon i < \rho]$ is residually small.*

Proof. Assume that R is an infinite ring and that $\rho \neq 0$ is a cardinal. Now let S be the multiplicative semigroup generated by $\{X_i \colon i < \rho\}$. Observe that every nonzero member f of $R[X_i \colon i < \rho]$ can be uniquely expressed in the form $r_0 + r_1\mathcal{X}_1 + \cdots + r_n\mathcal{X}_n$ for some $\mathcal{X}_1, \ldots, \mathcal{X}_n \in S$ with $\mathcal{X}_i \neq \mathcal{X}_j$ for $i \neq j$, and $r_0, \ldots, r_n \in R$, where $r_i \neq 0$ for $i > 0$. Let us call this the *canonical form* of f.

(1) Suppose that R is residually small, and let $f := r_0 + r_1\mathcal{X}_1 + \cdots + r_n\mathcal{X}_n \in R[X_i \colon i < \rho]$ be nonzero and in canonical form. If $n > 0$, then $\mathcal{X}_n$ has degree k for some integer $k > 0$. Now let $S(\mathcal{X}_n) \subseteq S$ be the set of divisors of $\mathcal{X}_n$ (if f has degree 0, then we put $S(\mathcal{X}_n) := \varnothing$). Since R is residually small and $r_n \neq 0$, there exists some ideal I of R such that $r_n \notin I$ and $|R/I| < |R|$. Now set

$$M := \{s_0 + s_1\mathcal{Y}_1 + \cdots + s_m\mathcal{Y}_m \colon m \geq 0, s_0 \in I, s_j \in R, \mathcal{Y}_j \in S, \text{ and if } \mathcal{Y}_j \in S(\mathcal{X}_n), \text{ then } s_j \in I\}.$$

It is straightforward to verify that M is an ideal of $R[X_i \colon i < \rho]$. Moreover, since $r_n \notin I$, it follows that $f \notin M$ (the condition "$s_0 \in I$" in the definition of M above is needed to establish that $f \notin M$ in case $n = 0$). Now, $S(\mathcal{X}_n)$ is finite; say $S(\mathcal{X}_n) = \{\mathcal{Y}_1, \ldots, \mathcal{Y}_a\}$. Define $\varphi \colon (R/I)^{a+1} \to R[X_i \colon i < \rho]/M$ by

$$\varphi(\overline{r}_0, \ldots, \overline{r}_a) := r_0 + r_1\mathcal{Y}_1 + \cdots + r_a\mathcal{Y}_a \pmod{M}.^3$$

It is not difficult to check that φ is both well-defined and surjective. Thus

$$|R[X_i \colon i < \rho]/M| \leq |(R/I)^{a+1}| < |R| \leq |R[X_i \colon i < \rho]|;$$

[3] if $n = 0$, then $\varphi \colon R/I \to R[X_i \colon i < \rho]$ is given by $\varphi(\overline{r}_0) := r_0 \pmod{M}$.

the strict inequality above follows from the fact that R is infinite, $|R/I| < |R|$, and $a+1 < \aleph_0$. This concludes the proof of (1).

(2) Suppose now that $R[X_i\colon i < \rho]$ is residually small and that $\rho \leq |R|$. Let $r \in R\backslash\{0\}$ be arbitrary. Since $r \in R[X_i\colon i < \rho\}$, there is an ideal M of $R[X_i\colon i < \rho]$ such that $r \notin M$ and $|R[X_i\colon i < \rho]/M| < |R[X_i\colon i < \rho]|$. Set $I := M \cap R$. Then it is easy to see (and well-known) that I is an ideal of R. Furthermore, $r \notin I$. Lastly, there is a canonical injection $\varphi\colon R/I \to R[X_i\colon i < \rho]/M$. We deduce that $|R/I| \leq |R[X_i\colon i < \rho]/M| < |R[X_i\colon i < \rho]| = |R|$.

(3) Assume that $\rho > |R|$, and let $f(X) \in R[X_i\colon i < \rho]$ be arbitrary. Then $f(X) \in R[X_{i_1}, \ldots, X_{i_n}]$ for some $i_1, \ldots, i_n < \rho$. Now let I be the ideal of $R[X_i\colon i < \rho]$ generated by $\{X_j\colon j \notin \{i_1, \ldots, i_n\}\}$. Then it is clear that $f(X) \notin I$. Moreover, we have $R[X_i\colon i < \rho]/I \cong R[X_{i_1}, \ldots, X_{i_n}]$. Therefore, $|R[X_i\colon i < \rho]/I| = |R[X_{i_1}, \ldots, X_{i_n}]| = |R| < |R[X_i\colon i < \rho]|$. The proof is now complete. □

Remark 3. *In the statement of* (2)*, one cannot dispense with the assumption that $\rho \leq |R|$. Indeed, the proof of Proposition 5 shows that if R any infinite ring and $\rho > |R|$, then $R[X_i\colon i < \rho]$ is residually small. Thus if R is an infinite field and $\rho > |R|$, then $R[X_i\colon i < \rho]$ is residually small, but R is not.*

Corollary 4. *Let R be a finite ring, and let ρ be a nonzero cardinal. Then $R[X_i\colon i < \rho]$ is residually small.*

Proof. It suffices by Theorem 3 to prove that $R[X]$ is residually small. To see this, let $f(X) \in R[X]$ be nonzero, and let n be an integer larger than the degree of f. Then $f(X) \notin \langle X^n \rangle$. Now, $R[X]/\langle X \rangle \cong R$; hence $R[X]/\langle X \rangle$ is finite. We apply Lemma 1 to conclude that $R[X]/\langle X^n \rangle$ is finite as well, and the proof is complete. □

We now move on to study direct products. As with polynomial extensions, the residually small property transfers nicely in this context.

Theorem 4. *Let $\{R_i\colon i \in I\}$ be a collection of rings. Then $R := \prod_{i\in I} R_i$ is residually small if and only if R is infinite and R_j is residually small for every $j \in I$ such that $|R_j| = |R|$.*

Proof. Suppose first that $R = \prod_{i\in I} R_i$ is residually small. Then by definition, R is infinite. Now suppose that $j \in I$ and $|R_j| = |R|$. Let $r_j \in R_j\backslash\{0\}$ be arbitrary. Define $(r_i) \in R$ by

$$r_i := \begin{cases} 0 & \text{if } i \neq j, \\ r_j & \text{if } i = j. \end{cases}$$

Since R is residually small, there exists an ideal I of R such that $(r_i) \notin I$ and $|R/I| < |R|$. Let $\pi_j\colon R \to R_j$ be projection onto the jth coordinate, and set $I_j := \pi_j(I)$. Then I_j is an ideal of R_j which does not contain r_j, lest $(r_i) \in I$. It remains to show that $|R_j/I_j| < |R_j|$. The map $\varphi\colon R/I \to R_j/I_j$ defined by $\varphi(I + r) := I_j + \pi_j(r)$ is easily seen to be a well-defined surjection. Thus $|R_j/I_j| \leq |R/I| < |R| = |R_j|$.

Conversely, assume that R is infinite and that R_j is residually small for all j such that $|R_j| = |R|$. Now let $(r_i) \in R\backslash\{0\}$ be arbitrary. Pick any j such that $r_j \neq 0$. If $|R_j| < |R|$, then define ideals I_i of the rings R_i by

$$I_i := \begin{cases} R_i & \text{if } i \neq j, \\ \{0\} & \text{if } i = j. \end{cases}$$

Then clearly $(r_i) \notin I := \prod_{i \in I} I_i$, and $R/I \cong R_j$. We deduce that $|R/I| = |R_j| < |R|$. Finally, suppose that $|R_j| = |R|$. Then by assumption, there exists an ideal I'_j of R_j such that $|R_j/I'_j| < |R_j|$ and $r_j \notin I'_j$. Set

$$I_i := \begin{cases} R_i & \text{if } i \neq j, \\ I'_j & \text{if } i = j. \end{cases}$$

It follows that $(r_i) \notin I := \prod_{i \in I} I_i$, and $R/I \cong R_j/I'_j$. Thus $|R/I| = |R_j/I'_j| < |R_j| = |R|$, and the proof is complete. $\square$

The following corollary is immediate.

Corollary 5. *Let I be an infinite index set and let $\{R_i : i \in I\}$ be a collection of finite rings. Then $\prod_{i \in I} R_i$ is residually small.*

Having presented several examples of residually small Noetherian rings, it is natural to enquire about the existence of residually small Artinian rings. The previous theorem settles this query.

Corollary 6. *There are no residually small Artinian rings.*

Proof. Suppose by way of contradiction that R is a residually small Artinian ring. As is well-known, $R = R_1 \times \cdots \times R_n$ for some Artinian local rings $R_1, \ldots, R_n$. By definition of residually small ring, R is infinite. Hence $|R| = |R_i|$ for some i, $1 \leq i \leq n$. Theorem 4 implies that R_i is residually small, and hence R_i possesses a proper large ideal. However, Proposition 8 of [16] states that an infinite Artinian local ring does not possess a proper large ideal. This contradiction completes the proof. $\square$

Having studied ring constructions which respect residual smallness quite nicely, we now study other constructions for which the property is not as well-behaved. First on our list is quotient rings. Recall that a subset S of a ring R is a *multiplicative set* if S is closed under multiplication, $1 \in S$, and $0 \notin S$. One then defines the *quotient ring* of R relative to the multiplicative set S by $R_S := \{\frac{r}{s} : r \in R, s \in S\}$, with canonical addition and multiplication and equality defined as follows: $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ if and only if there is $s \in S$ with $s(s_2 r_1 - s_1 r_2) = 0$.

Assume now that R is a domain. Then observe that $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ if and only if $s_2 r_1 = s_1 r_2$. It follows that the map $x \mapsto \frac{x}{1}$ is an embedding of R into R_S. Note further that if $S = \{1\}$, then $R_S \cong R$. At the other extreme, if $S = R\backslash\{0\}$, then R_S is the quotient field of R. We shall make use of the following standard results on quotient rings.

Lemma 5 ([6], Theorem 4.4, Proposition 5.8, and Corollary 5.9)**.** *Let R be a ring, and let $S \subseteq R$ be multiplicative. Then*

(1) *Every proper ideal of R_S is of the form $I^e := \{\frac{i}{s} : i \in I, s \in S\}$ for some ideal I of R which is disjoint from S.*

(2) *If I is an ideal of R disjoint from S, then $R_S/I^e \cong (R/I)_{(S+I)/I}$.*
(3) *Suppose P is a prime ideal of R and $S = R\backslash P$. Setting $R_P := R_{R\backslash P}$, we have $R_P/P^e \cong F$, where F is the quotient field of R/P.*

The residual smallness property can easily be shown to "travel downward" from a quotient ring R_S to the base ring R if R is a domain. More generally,

Proposition 6. *Suppose that $R \subseteq S$ is an extension of rings such that S is residually small and $|R| = |S|$. Then R is residually small.*

Proof. The proof is analogous to the proof of (2) of Theorem 3, and is left to the reader. □

Conversely, residual smallness does not, in general, "pass upward" to quotient rings. Indeed, if D is a residually small domain, then the quotient field K of D is a field, hence not residually small. In fact, there exist residually small domains D and multiplicative sets $S \subseteq D$ such that D_S is not a field, yet D_S is also not residually small, as the following simple example verifies.

Example 2. *The ring $\mathbb{Z}[X]$ is residually small, but the localization $\mathbb{Z}[X]_{\langle X\rangle}$ is not.*

Proof. It follows immediately from Theorem 3 that $\mathbb{Z}[X]$ is residually small. Further, Lemma 5 implies that the residue field of the local ring $\mathbb{Z}[X]_{\langle X\rangle}$ is (isomorphic to) $\mathbb{Q}$. Finally, we conclude from Corollary 1 that $\mathbb{Z}[X]_{\langle X\rangle}$ is not residually small. □

We now present a sufficient condition for a quotient ring of a domain to be residually small.

Theorem 5. *Let D be an infinite domain, and let $S \subseteq D$ be multiplicative. Then the following conditions are sufficient for D_S to be residually small:*
(1) *S is a proper subset of $D\backslash\{0\}$, and*
(2) *for every nonzero $d \in D\backslash S$, there exists an ideal I of D disjoint from S such that $Sd \cap I = \varnothing$ and $|D/I| < |D|$.*

Proof. Let D be an infinite ring and $S \subseteq D$ be multiplicative. Suppose that (1) and (2) above hold, and let $\frac{x}{s} \in D_S$ be nonzero. To prove that D_S is residually small, it suffices to find a large ideal of D_S which does not contain x. We consider two cases.

Case 1. $x \in S$. By (1), there exists some nonzero $d \in R\backslash S$. Then (2) furnishes us with an ideal I of D disjoint from S such that $|D/I| < |D|$. We claim that $x \notin I^e$. If so, then $xs \in I$ for some $s \in S$. But $x \in S$ and S is multiplicative. Hence $xs \in I \cap S$, contradicting that I is disjoint from S. It remains to show that $|D_S/I^e| < |D_S|$. By (2) of Lemma 5, we have $|D_S/I^e| = |(D/I)_{(S+I)/I}| \leq |D/I|^2 < |D| = |D_S|$.

Case 2. $x \notin S$. By (2), there is an ideal I of D disjoint from S such that $Sx \cap I = \varnothing$ and $|D/I| < |D|$. The remainder of the proof proceeds as in Case 1. □

We now study how well factor rings respect residual smallness. To begin, note trivially that there does not exist a ring R with the property that R/I is residually small for all proper ideals I of R. Indeed, if J is any maximal ideal of R, then R/J is a field, hence not residually small. Less generally, one may hope to prove that if R is residually small and I is an ideal of R such that R/I

is not a field and $|R/I| = |R|$, then R/I is residually small. This can fail too, even in the case where R is a semilocal Noetherian integral domain. To prove this, we shall require the following lemma.

Lemma 6 ([11], Theorem 2.6)**.** *Let $(\rho, \kappa_1, \ldots, \kappa_n)$ be a sequence of cardinals where each κ_i is a prime power or is infinite such that*

$$\kappa_i + \aleph_0 \leq \rho \leq \kappa_i^{\aleph_0}$$

for each i, $1 \leq i \leq n$. Then there is a principal ideal domain D of cardinality ρ with exactly n maximal ideals $J_1, \ldots, J_n$ such that $|D/J_i| = \kappa_i$ for $1 \leq i \leq n$.

Example 3. *Suppose that ρ is an infinite cardinal such that there is some cardinal κ with $\kappa < \rho \leq \kappa^{\aleph_0}$. Then there exists a residually small principal ideal domain D with exactly two maximal ideals J_1 and J_2 such that for every proper ideal I of D not contained in J_1: $|D/I| = |D|$, yet D/I is not residually small.*

Proof. Let ρ and κ be as stated. Of course, we may assume that if κ is finite, then κ is prime. By Lemma 6, there exists a principal ideal domain D of cardinality ρ with exactly two maximal ideals J_1 and J_2 such that $|D/J_1| = \kappa$ and $|D/J_2| = |D|$. We apply Theorem 2 to conclude that D is residually small. Now suppose that I is a proper ideal of D not contained in J_1. Then $I \subseteq J_2$, and we see that $|D| = |D/J_2| \leq |D/I| \leq |D|$; hence equality holds throughout. It remains to show that D/I is not residually small. If D/I is residually small, then (since D/I is local with maximal ideal J_2/I) J_2/I is a large ideal of D/I. But $(D/I)/(J_2/I) \cong D/J_2$. We deduce that $|(D/I)/(J_2/I)| = |D/J_2| = |D| = |D/I|$, showing that J_2/I is not large. This contradiction concludes the proof. □

Despite the previous example, we do have the following general result.

Proposition 7. *Suppose that R is a Noetherian local ring and that I is an ideal of R such that $|R/I| = |R|$. Then R is residually small if and only if R/I is residually small.*

Proof. Let R be a Noetherian local ring with maximal ideal J, and suppose that I is an ideal of R such that $|R/I| = |R|$. Then R/I is also a Noetherian local ring with maximal ideal J/I. Further, $|(R/I)/(J/I)| = |R/J|$. Thus $|R/J| < |R|$ if and only if $|(R/I)/(J/I)| < |R/I|$. We now invoke Corollary 1 to complete the proof. □

Remark 4. *If R is a residually small Noetherian local ring and I is an ideal of R such that $|R/I| < |R|$, it does not follow that R/I is residually small, even if R/I is infinite. This can be seen by taking $R := \mathbb{Q}[[T]]$ and $I := \langle T \rangle$. Further, it is not hard to construct an example of a residually small Noetherian local ring R with a nonzero ideal I such that $|R/I| = |R|$. Indeed, let k be a countable field, and let $R := k[[T_1, \ldots, T_n]]$ be the power series ring over k in the $n > 1$ variables $T_1, \ldots, T_n$. Then $|R| = 2^{\aleph_0}$ and $R/\langle T_1, \ldots, T_n \rangle \cong k$, hence is countable; thus R is residually small by Corollary 1. Now choose $I := \langle T_i \rangle$, where $1 \leq i \leq n$ is arbitrary. It is easy to see that $|R/I| = |k[[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n]]| = 2^{\aleph_0} = |R|$.*

We conclude the section with an investigation of the behavior of residual smallness in integral extensions. Recall that a ring extension S is *integral over the ring R* provided every $s \in S$ is the root of a monic polynomial $f(X) \in R[X]$. As we showed in Proposition 6, residual smallness passes easily from top to bottom if $|R| = |S|$. However, it does not always pass upward to integral extensions. To simplify our analysis, we restrict our study of residual smallness to extensions of domains. Our next example shows that, as claimed, residual smallness does not always travel upward, even for integral extensions of Noetherian domains.

Example 4. *There exists an integral extension $D_1 \subseteq D_2$ of Noetherian domains such that D_1 is residually finite but D_2 is not residually small.*

Proof. Let $\mathbb{F}$ be a finite field, and let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$. Then of course, $\overline{\mathbb{F}}$ is integral over $\mathbb{F}$. Hence $\overline{\mathbb{F}}[X]$ is integral over $\mathbb{F}[X]$ (cf. Gilmer [6], Theorem 10.7). Corollary 4 tells us that $\mathbb{F}[X]$ is residually small; since $\mathbb{F}[X]$ is countable, we conclude that $\mathbb{F}[X]$ is residually finite. However, for any proper ideal I of $\overline{\mathbb{F}}[X]$, the map $r \mapsto I + r$ is an injection of $\overline{\mathbb{F}}$ into the countably infinite ring $\overline{\mathbb{F}}[X]/I$. It follows that $\overline{\mathbb{F}}[X]$ is not residually small. □

Observe that the domain D_2 in the above example is not finitely generated over D_1. We show that if this is the case, then residual smallness survives.

Proposition 8. *Let D_1 be a residually small Noetherian domain, and suppose D_2 is a finite ring extension of D_1 (that is, D_2 is finitely generated as a D_1-module). Then D_2 is also residually small.*

Proof. Let $D_1 \subseteq D_2$ be a finite extension of domains and suppose that D_1 is Noetherian and residually small. Since D_2 is the homomorphic image of $D_1[X_0, \ldots, X_n]$ for some $n \in \mathbb{N}$, it follows that D_2 is Noetherian. It suffices by Theorem 2 to prove that D_2 has a proper large ideal. Because D_1 is residually small, D_1 has a proper large ideal I. Let J_1 be a maximal ideal of D_1 containing I. Then J_1 is large as well. Finite extensions are integral ([6], Theorem 9.3), and thus D_2 is integral over D_1. Hence there exists a maximal ideal J_2 of D_2 such that $J_2 \cap D_1 = J_1$ ([6], Theorems 11.4 and 11.5). As we have seen, there is an injective (ring) map $\varphi\colon D_1/J_1 \to D_2/J_2$. Identifying D_1/J_1 with its image in D_2/J_2, it follows that D_2/J_2 is a finite field extension of D_1/J_1. Thus $|D_2/J_2| = |D_1/J_1|^n$ for some positive integer n. Therefore $|D_2/J_2| = |D_1/J_1|^n < |D_1| = |D_2|$, and the proof is complete. □

Corollary 7. *Suppose that D is a residually small Noetherian domain and that I is an ideal of $D[X]$ containing a monic polynomial such that $I \cap D = \{0\}$. Then $D[X]/I$ is residually small.*

Proof. Let D and I have the above properties. Since $I \cap D = \{0\}$, it follows that the map $d \mapsto I + d$ maps D injectively into $D[X]/I$; thus (up to isomorphism) $D[X]/I$ is a ring extension of D. Now let $f \in I$ be monic of degree n. Then it is easy to see that $\{I + 1, I + X, \ldots, I + X^{n-1}\}$ generates $D[X]/I$ over D. We now apply Proposition 8. □

Also of interest is the fact that the domain D_1 constructed in Example 4 is countable. Indeed, this had to be so, as we now demonstrate.

Proposition 9. *Suppose that $D_1 \subseteq D_2$ is an integral extension of domains such that D_1 is uncountable and D_2 is Noetherian. If D_1 is residually small, then so is D_2.*

Sketch of Proof. Suppose D_1 and D_2 are as stated and that D_1 is residually small. The proof proceeds almost identically to the proof of Proposition 8; as such, we indicate only the necessary modifications to the previous proof. By Lemma 10.1 of [6], D_2/J_2 is integral over D_1/J_1. But both rings are fields, and hence D_2/J_2 is algebraic over D_1/J_1. Therefore (by the standard set-theoretic argument counting roots of polynomials over fields), $|D_2/J_2| \leq \max(\aleph_0, |D_1/J_1|) < |D_1| \leq |D_2|$, concluding the proof. □

Given our results thus far, the next natural question is as follows: suppose that D_1 is an uncountable residually small domain and that $D_1 \subseteq D_2$ is an integral extension. Must D_2 be residually small as well? While we currently do not know the answer to this question, we do have partial results. Toward this end, we introduce a new definition.

Definition 2. *Let R be an infinite ring. Say that R is strongly residually small if for every $r \in R \backslash \{0\}$, there exists a prime ideal P of R such that $r \notin P$ and $|R/P| < |R|$.*

Note trivially that a strongly residually small domain is not a G-domain[4]: indeed, if D is a domain for which the interesection of the nonzero prime ideals of D is nonzero, then choose a nonzero d in the intersection. Clearly there can be no prime ideal P of D such that $d \notin P$ and $|D/P| < |D|$. We conclude that any strongly residually small domain has infinitely many prime ideals. Thus, for example, no valuation domain of finite Krull dimension is strongly residually small. Applying Corollary 1, a discrete valuation domain $(V, \mathfrak{m})$ is residually small if and only if $|V/\mathfrak{m}| < |V|$. This yields a nontrivial class of examples of residually small domains which are not strongly residually small. On the other hand, it is not hard to show that there exist strongly residually small domains of every infinite cardinality, as the next example demonstrates.

Example 5. *The following hold:*

(1) *Let F be a field. Then $F[X]$ is strongly residually small if and only if F is finite.*
(2) *Let $\rho > \aleph_0$ and D be an infinite domain of size less than ρ. Then $D[X_i \colon i < \rho]$ has size ρ and is strongly residually small.*

Proof. (1) If F is an infinite field, then $|F[X]| = |F|$. The argument presented in the proof of Example 4 shows that $F[X]$ is not residually small. Now suppose that F is a finite field, and let $f(X) \in F[X]$ be nonzero. Choose any irreducible polynomial $p(X)$ which does not divide $f(X)$. Then $f(X) \notin \langle p(X) \rangle$, $\langle p(X) \rangle$ is a maximal (hence prime) ideal, and $F[X]/\langle p(X) \rangle$ is finite.

(2) Assume that $\rho > \aleph_0$ and D is an infinite domain of size less than ρ. That $|D[X_i \colon i < \rho]| = \rho$ follows from elementary set theory. Now let $f(X) \in D[X_i \colon i < \rho]$ be nonzero. Without loss of generality, $f(X) \in D[X_0, \ldots, X_n]$ for some natural number n. Then $f(X) \notin \langle X_i \colon n < i < \rho \rangle := P$. Moreover, $|D[X_i \colon i < \rho]/P| = |D| < \rho$. □

We now prove that residual smallness passes to integral extensions when the base ring is strongly residually small.

Proposition 10. *Let D_1 be an uncountable strongly residually small domain, and let $D_1 \subseteq D_2$ be an integral extension. Then D_2 is strongly residually small.*

[4] Recall that a domain D is a G-domain if the intersection of the nonzero prime ideals of D is nonzero.

Proof. Suppose $D_1 \subseteq D_2$ is an integral extensions and D_1 is uncountable and strongly residually small. Now let $x \in D_2 \backslash \{0\}$ be arbitrary. Then of course, the principal ideal $D_2 x$ is nonzero. Since D_2 is integral over D_1, it follows that $D_2 x \cap D_1$ is a nonzero ideal of D_1 ([6], Lemma 11.1). Hence there exists some $d \in D_2$ such that dx is a nonzero element of D_1. As D_1 is strongly residually small, there exists a prime ideal $P_1 \subseteq D_1$ such that $dx \notin P_1$ and $|D_1/P_1| < |D_1|$. By the Lying Over Theorem, there exists a prime ideal $P_2 \subseteq D_2$ such that $P_2 \cap D_1 = P_1$. We deduce that $dx \notin P_2$, and thus $x \notin P_2$. Now, D_2/P_2 is integral over D_1/P_1. Let S be the set of nonzero elements of D_1/P_1. Then by Proposition 10.2 of [6], the quotient ring $(D_2/P_2)_S$ is integral over $(D_1/P_1)_S$. Observe that $(D_1/P_1)_S$ is the quotient field of D_1/P_1. Thus $(D_2/P_2)_S$ is integral over a field, hence is also a field. Finally, $|D_2/P_2| = |(D_2/P_2)_S| \leq \max(\aleph_0, |(D_1/P_1)_S|) = \max(\aleph_0, |D_1/P_1|) < |D_1| \leq |D_2|$, and the proof is complete. ☐

Remark 5. *Recall from Example 5 that if F is a finite field, then $F[X]$ is a countable strongly residually small domain. Thus Example 4 shows that we cannot eliminate the assumption that D_1 is uncountable in the statement of the previous proposition.*

We will soon show that for many infinite cardinals κ, all residually small domains of size κ are strongly residually small. Before proceeding, we pause to recall some basic set theory. Let α be a nonzero limit ordinal, and suppose S is a subset of α (that is, S is a set of ordinals and every member of S is smaller than α). Then S is *cofinal* in α if $\cup S = \alpha$. The *cofinality* of α, denoted $cf(\alpha)$, is the smallest cardinality of a cofinal subset of α. An infinite cardinal κ is *regular* if $cf(\kappa) = \kappa$. It is well-known that $cf(\alpha)$ is a regular cardinal for every nonzero limit ordinal α. For further reading on set theory, we refer the reader to the standard text Jech [10].

Proposition 11. *Suppose that κ is a cardinal with the following two properties:*

(1) *κ has uncountable cofinality, and*
(2) *$\beta^{\aleph_0} < \kappa$ for all cardinals $\beta < \kappa$.*

Then every residually small domain of size κ is strongly residually small.

Proof. Let κ satisfy (1) and (2) above, and suppose that D is a residually small domain of cardinality κ. Now let $x \in D$ be arbitrary. Since D is residually small (and a domain), it follows that for every positive integer n, there exists an ideal I_n such that $x^n \notin I_n$ and $\kappa_n := |D/I_n| < \kappa$. Let $\rho := \bigcup_{n \in \mathbb{Z}^+} \kappa_n$. Since the cardinal κ has uncountable cofinality, we deduce that $\rho < \kappa$. By (2), also $\rho^{\aleph_0} < \kappa$. Setting $I^* := \bigcap_{n=1}^{\infty} I_n$, observe that D/I^* maps injectively into $\prod_{n=1}^{\infty} D/I_n$. Therefore,

$$|D/I^*| \leq \left| \prod_{n=1}^{\infty} D/I_n \right| = \prod_{n=1}^{\infty} \kappa_n \leq \rho^{\aleph_0} < \kappa = |D|.$$

Since $x^n \notin I_n$ for each n, we see by definition of I^* that $x^n \notin I^*$ for every positive integer n. Hence I^* is an ideal which is disjoint from the multiplicative set $S := \{x^n \colon n > 0\}$. By Zorn's Lemma, I^* is contained in an ideal P which is maximal with respect to being disjoint from S. As is well-known, P is a prime ideal. Finally, we see that $x \notin P$ and $|D/P| \leq |D/I^*| < |D|$. The proof is concluded. ☐

We now establish an abundance of cardinals κ satifying (1) and (2) of the previous proposition. Recall that the *beth cardinals* are defined by recursion on the ordinals as follows: $\beth_0 := \aleph_0$, $\beth_{a+1} := 2^{\beth_a}$, and $\beth_b := \bigcup_{i<b} \beth_i$ if b is a nonzero limit. It is an exercise in elementary set theory to show that if b is a limit ordinal of uncountable cofinality (for example, if b is an uncountable regular cardinal), then $\beth_b$ satisfies properties (1) and (2) above. Thus there are arbitrarily large cardinals κ which satisfy (1) and (2).

We end this section by providing an example of a residually small ring-theoretic statement that is independent of ZFC. First we recall a bit more set theory. The *continuum hypothesis* (CH) is the assertion that there is no cardinal number κ satisfying $\aleph_0 < \kappa < 2^{\aleph_0}$ (equivalently, every infinite set of real numbers is either countable or in one-to-one correspondence with $\mathbb{R}$). The *generalized continnum hypothesis* (GCH) is the assertion that for every infinite cardinal β, there is no cardinal κ satisfying $\beta < \kappa < 2^{\beta}$. It is well-known (through the work of Gödel and Cohen) that CH and GCH can neither be proved nor refuted from the usual axioms of ZFC (that is, CH and GCH are independent of ZFC).

In the presence of GCH, cardinal exponentiation becomes much more tame. In particular,

Fact 1 ([10], Theorem 5.15). *Assume that GCH holds, and suppose that κ and λ are infinite cardinals such that $\kappa < cf(\lambda)$. Then $\lambda^{\kappa} = \lambda$.*

Returning to ring theory, recall that a ring R is *semiprimitive* if the Jacobson radical of R is trivial. An integral domain D is *one-dimensional* (Krull dimension is what is intended here) if D is not a field and every nonzero prime ideal of D is maximal. We now present the final theorem of this note.

Theorem 6. *For $i \in \{0, 1, 2\}$, let φ_i denote the statement, "All one-dimensional residually small domains of cardinality $\aleph_i$ are semiprimitive." Then φ_0 and φ_1 can be refuted in ZFC, but φ_2 is independent of ZFC.*

Proof. Observe that $2 + \aleph_0 \leq \aleph_0 < \aleph_1 \leq 2^{\aleph_0}$. Lemma 6 furnishes us with local principal ideal domains D_0 and D_1 of cardinality $\aleph_0$ and $\aleph_1$, respectively, such that $|D_i/J_i| = 2$, where J_i is the maximal ideal of D_i. Therefore, D_0 and D_1 are one-dimensional, and Corollary 1 implies that D_0 and D_1 are residually small. Since D_0 and D_1 are local and not fields, clearly D_0 and D_1 are not semiprimitive.

Suppose now that CH fails. Then $2+\aleph_0 < \aleph_2 \leq 2^{\aleph_0}$. It now follows from the above argument that there exists a one-dimensional residually small domain of size $\aleph_2$ which is not semiprimitive. On the other hand, suppose that GCH holds, and let D be a one-dimensional residually small domain of size $\aleph_2$. We shall prove that D is semiprimitive. Since $\aleph_2$ is a successor cardinal, $\aleph_2$ is regular (this can be proved in ZFC). Let λ be a cardinal such that $\lambda < \aleph_2$. Then $\lambda \leq \aleph_1$. Employing Fact 1, we see that $\lambda^{\aleph_0} \leq \aleph_1^{\aleph_0} = \aleph_1 < \aleph_2$. The hypotheses of Proposition 11 are now satisfied, and we deduce that D is strongly residually small. Let $d \in D\backslash\{0\}$ be arbitrary. Then there is a large prime ideal P of D such that $d \notin P$. Since D is one-dimensional, P is maximal. It follows that $J(D) = \{0\}$, and D is semiprimitive, as claimed. $\square$

4. Open questions

We end the paper with some open questions for further study.

Open Question 1. *We showed in Proposition 7 that if R is a Noetherian local ring and I is an ideal of R with $|R/I| = |R|$, then R is residually small if and only if R/I is residually small. Does this result still hold if we omit the word "Noetherian"?*

Open Question 2. *Suppose that $D_1 \subseteq D_2$ is an integral extension of domains and that D_1 is residually small and uncountable. Must D_2 be residually small? Is this question even decidable in ZFC?*

Open Question 3. *Suppose that $D_1 \subseteq D_2$ is a finite ring extension and D_1 is residually small. Must D_2 be residually small?*

Open Question 4. *Let D be an infinite domain, and suppose $S \subseteq D$ is multiplicative. Are the sufficient conditions given in Theorem 5 for the quotient ring D_S to be residually small also necessary?*

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