RESIDUAL SMALLNESS IN COMMUTATIVE ALGEBRA

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Abstract. In Oman & Salminen [19], the authors introduce and study residually small rings, defined as follows: an infinite commutative ring R with identity is *residually small* if for every $r \in R\backslash\{0\}$, there exists an ideal I_r of R such that $r \notin I_r$ and $|R/I_r| < |R|$. The purpose of this note is to extend our study. In particular, we continue our investigation of residually small rings and then generalize this notion to modules.

1. Introduction

Recall that a group G is *residually finite* if for every $g \in G\backslash\{e\}$, there exists a normal subgroup H_g of G such that $g \notin H_g$ and G/H_g is finite. There is an extensive literature on residually finite groups; we refer the interested reader to Hartley [5], Magnus [13], and Segal [21] for some surveys on the subject.

In Lewin [12] and Orzech & Ribes [20], the concept of residual finiteness is ported over to rings in the natural way. Specifically, an infinite ring R is called *residually finite* if for every $r \in R\backslash\{0\}$, there exists an ideal I_r of R such that $r \notin I_r$ and R/I_r is finite. In these papers, the authors translate many of the previous results on residually finite groups to rings. Much later, Varadarajan ([23]–[24]) extends the notion of residual finiteness to modules.

Most recently ([19]), we extend the notion of residual finiteness of rings to other cardinalities, calling such rings *residually small*. The precise definition is as follows (throughout this paper, as is customary, we shall denote the cardinality of a set X by $|X|$):

Definition 1. *Let R be an infinite commutative ring with identity. Then R is residually small if for every nonzero $r \in R$, there exists an ideal I_r of R not containing r such that $|R/I_r| < |R|$.*

This definition generalizes the notion of residual finiteness in that all residually finite rings are residually small. Definition 1 is also closely related to the concept of a homomorphically smaller module. In Oman & Salminen [18] the authors, borrowing terminology introduced in Tucci [22], define an infinite, unitary module M over a commutative ring R to be *homomorphically smaller* (HS for short) if and only if $|M/N| < |M|$ for every nonzero submodule N of M. We also define a ring R to be an *HS ring* if R is an HS module over itself, that is, if R is infinite and $|R/I| < |R|$ for every nonzero ideal I of R. This definition extends other notions of residual finiteness which also appear in the literature (unfortunately, the terminology 'residually finite ring' is multiply defined). For instance, in Chew & Lawn [1], the authors call a (not necessarily commutative) ring R residually finite if R/I is finite for every nonzero ideal I of R. In Levitz and Mott [11], the

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authors extend Chew and Lawn's results to rings without identity and say such rings have the *finite norm property*. This concept was also considered by Ion, Militaru, and Niţă in [7] and [8], where they use the nomenclature *rings with finite quotients*.

The purpose of this paper is to continue the investigations initiated in [19]. In the next section, we continue our study of residually small rings by considering monoid rings. Then in Section 3, we introduce the notion of residual smallness for modules. In the final section, we close with some open questions. We conclude the introduction by mentioning that *all monoids in subsequent sections are assumed commutative, all rings are assumed commutative with* $1 \neq 0$*, and all modules and ring extensions are assumed to be unitary unless stated otherwise.*

2. Residual smallness in monoid rings

In Section 3 of [19], we studied how well the residually small property (for rings) behaves relative to polynomial ring extensions, direct products, quotient rings, factor rings, and integral extensions. The purpose of this section is to continue our investigation in the context of monoid rings. First, we review some terminology.

Let S be an additive monoid. Then S is *torsion-free* if for any positive integer n and any $x, y \in S$: $nx = ny$ implies $x = y$; S is *cancellative* if for all $a, b, c \in S$: if $a + b = a + c$, then $b = c$. Next, suppose R is a ring and let S be an additive monoid. Then the *monoid ring* $R[S]$ is the collection of all functions $f : S \to R$ which are finitely nonzero. Functions in $R[S]$ are added componentwise and multiplication is given as follows: $fg(s) = \sum_{a+b=s} f(a)g(b)$. It will be convenient to express $f \in R[S]$ by $r_1x^{s_1} + \ldots + r_nx^{s_n}$, where the s_i's are distinct (this represents the function $f : S \to R$ that is 0 outside $\{s_1, \ldots, s_n\}$ and such that $f(s_i) = r_i$ for $1 \leq i \leq n$). Now, if S is a monoid, then a (possibly empty) set $I \subseteq S$ is an *ideal* of S provided $x \in I$ and $s \in S$ imply that $s + x \in I$. For an ideal I of S, the *Rees congruence* of I is defined on S by $a \sim b \Leftrightarrow a = b$ or $a, b \in I$. It is well-known that $\sim$ is a congruence on S. The quotient $S/\sim := S/I$ inherits a natural addition from S, called the *Rees quotient of S by I*. Extending the notion of residual smallness to monoids, let us say that an infinite monoid S is residually small if for all $s \in S$, there exists an ideal I of S not containing s such that $|S/I| < |S|$.

We shall require the following results from the literature.

Lemma 1 ([19], Proposition 7)**.** *Suppose that $R \subseteq S$ is an extension of rings such that S is residually small and $|R| = |S|$. Then R is residually small.*

Lemma 2 ([4], Theorems 7.7 and 8.1, respectively)**.** *Let R be a ring and let S be a monoid. Then the following hold:*

(1) *$R[S]$ is Noetherian if and only if R is Noetherian and S is finitely generated.*
(2) *$R[S]$ is an integral domain if and only if R is an integral domain and S is torsion-free and cancellative.*

Lemma 3 ([18], Theorem 2)**.** *Let D be an infinite Noetherian domain. Then D is residually small if and only if there is a proper ideal I of D such that $|D/I| < |D|$.*

We are now ready to prove the first theorem of this section.

Theorem 1. *Let R be a ring and let S be an additive monoid. Then the following hold:*

(1) *If R is a residually small Noetherian domain and S is finitely generated, torsion-free and cancellative, then $R[S]$ is residually small.*
(2) *If R and S are residually small, then $R[S]$ is residually small.*
(3) *If $|R| \neq |S|$ and the larger of the two is residually small, then $R[S]$ is residually small.*
(4) *If $R[S]$ is residually small and $|S| \leq |R|$, then R is residually small.*

Proof. Assume that R and S are as stated.

(1) Assume that R is a residually small Noetherian domain and S is finitely generated, torsion-free, and cancellative. Then it follows immediately from Lemma 2 that $R[S]$ is a Noetherian domain. To prove that $R[S]$ is residually small, it suffices by Lemma 3 to prove that there is a proper ideal I of $R[S]$ such that $|R[S]/I| < |R[S]|$. Since R is residually small, there is a proper ideal J of R such that $|R/J| < |R|$. Now define $\varphi\colon R[S] \to R/J$ by $\varphi(r_1x^{s_1} + \cdots + r_nx^{s_n}) := \overline{r}_1 + \cdots + \overline{r}_n$. One checks at once that φ is a well-defined ring surjection; let K be the kernel. Then we see that $|R[S]/K| = |R/J| < |R| \leq |R[S]|$, and this completes the proof of (1).

(2) Suppose that R and S are residually small, and consider an arbitrary nonzero element $f := r_1x^{s_1} + \cdots + r_nx^{s_n} \in R[S]$. We may assume that $r_1 \neq 0$. As R is residually small, there exists an ideal I_R of R not containing r_1 such that $|R/I_R| < |R|$. Similarly, there exists an ideal I_S of S not containing s_1 such that $|S/I_S| < |S|$. Now set

$$J := \{a_1x^{b_1} + \cdots + a_mx^{b_m} \colon \text{ for } 1 \leq i \leq m,\ a_i \in R,\ b_i \in S, \text{ and either } a_i \in I_R \text{ or } b_i \in I_S\}.$$

It is easy to verify that J is an ideal of $R[S]$ and that $f \notin J$. Next, consider the monoid ring $\mathfrak{R} := (R/I_R)[S/I_S]$. We claim that

$$|\mathfrak{R}| < |R[S]|. \tag{2.1}$$

This is clearly true if both R/I_R and S/I_S are finite. Now assume that one of R/I_R and S/I_S is infinite. Then $|\mathfrak{R}| = \max(|R/I_R|, |S/I_S|) < \max(|R|, |S|) = |R[S]|$, and (2.1) is established. Next, define $\varphi\colon \mathfrak{R} \to R[S]/J$ by $\varphi(\overline{a}_1x^{\overline{b}_1} + \cdots + \overline{a}_mx^{\overline{b}_m}) := a_1x^{b_1} + \cdots + a_mx^{b_m} \bmod(J)$. We must show that φ is well-defined. Toward this end, assume that $\overline{a}_1x^{\overline{b}_1} + \cdots + \overline{a}_mx^{\overline{b}_m} = \overline{c}_1x^{\overline{d}_1} + \cdots + \overline{c}_kx^{\overline{d}_k}$ where the $\overline{b}_i$ and $\overline{d}_j$ are distinct and each $\overline{a}_i$, $\overline{c}_j$ is nonzero (the case where both elements are zero is trivial to handle). Then $k = m$, and we may assume (reordering if necessary) that for $1 \leq i \leq m$, both $\overline{a}_i = \overline{c}_i$ and $\overline{b}_i = \overline{d}_i$. To finish the proof, it suffices to show that $a_ix^{b_i} - c_ix^{d_i} \in J$ for $1 \leq i \leq m$. Clearly it suffices to prove the assertion for $i = 1$. Thus consider $a_1x^{b_1} - c_1x^{d_1}$. Since $\overline{b}_1 = \overline{d}_1$, it follows from definition that either $b_1, d_1 \in I_S$ or $b_1 = d_1$. In the former case, $a_1x^{b_1} - c_1x^{d_1} \in J$ by definition of J. Now suppose that $b_1 = d_1$. Since $\overline{a}_1 = \overline{c}_1$, we have $a_1 - c_1 \in I_R$. Therefore $a_1x^{b_1} - c_1x^{d_1} = (a_1 - c_1)x^{b_1} \in J$. This proves that φ is well-defined. Now φ is clearly onto, and invoking (2.1), $|R[S]/J| \leq |\mathfrak{R}| < |R[S]|$. Hence $R[S]$ is residually small.

(3) The proof is analogous to the proof of (2) (the notation will be the same). As such, we sketch the argument, but eliminate details. Suppose first that $|S| < |R|$ and R is residually small. Now

set $I_S := \varnothing$. The remainder of the proof goes through unchanged. Similarly, suppose that $|R| < |S|$ and S is residually small. Then choose $I_R := \{0\}$.

(4) Immediate from Lemma 1. $\square$

We shall soon establish several corollaries of the previous theorem. First, we require an additional definition and a lemma. Let S be an additive monoid and let $s \in S$. Then a *divisor* of s is an element $d \in S$ such that $d + x = s$ for some $x \in S$; if this is the case, we write $d|s$. In what follows, for $s \in S$, we denote the set of divisors of s by $D(s)$.

Lemma 4. *Suppose that S is an infinite monoid such that for every $s \in S$, $|D(s)| < |S|$. Then S is residually small.*

Proof. Let S be as stated and choose $s \in S$ arbitrarily. Set $I := \{x \in S \colon x \text{ is not a divisor of } s\}$. Then it is immediate that $s \notin I$ and that I is an ideal of S. Moreover, $|S/I| = |D(s)| + 1 < |S|$. $\square$

Corollary 1 ([19], (1) of Theorem 3)**.** *If R is a residually small ring and ρ is a nonzero cardinal, then the polynomial ring $R[X_i \colon i < \rho]$ in ρ indeterminates is residually small.*

Proof. Since $R[X_i \colon i < \rho] \cong R[S]$, where $S = \bigoplus_\rho \mathbb{N}$, it suffices by (2) of Theorem 1 and Lemma 4 to establish that $D(s)$ is finite for every $s \in S$. This is patent, and the proof is complete. $\square$

Corollary 2. *The following hold:*

(1) *If S is a monoid, then there exists a ring R such that $R[S]$ is residually small.*
(2) *If R is a ring, then there exists a monoid S such that $R[S]$ is residually small.*

Proof. If S is a monoid, simply choose any residually small ring larger than S (such a ring is guaranteed by Corollary 1). Apply (3) of Theorem 1.

Now let R be a ring and let κ be an infinite cardinal larger than $|R|$. Lemma 2.7 of [18] furnishes us with a totally ordered abelian group G with the property that for every $g \geq 0$, the set $\{x \in G \colon 0 \leq x \leq g\}$ has cardinality less than κ. Let $S := \{g \in G \colon g \geq 0\}$ be the non-negative cone of G and let $s \in S$ be arbitrary. Then $|D(s)| < |S|$. Applying (3) of Theorem 1 and Lemma 4 concludes the proof. $\square$

Next, we discuss some of the hypotheses of Theorem 1. We showed in (2) of Theorem 1 that the residual smallness of a ring R and monoid S implies the residual smallness of the monoid ring $R[S]$. If we assume only that S is residually small, then $R[S]$ need not be residually small. In fact, there is a sense in which the residual smallness of $R[S]$ can fail as badly as possible: take R to be an infinite field and $S := \mathbb{N}$. Then of course, $R[S] \cong R[X]$. Moreover, if I is any proper ideal of $R[X]$, then there is a natural injection of R into $R[X]/I$. Hence $|R[X]/I| = |R[X]|$ for every proper ideal I of R. On the other hand, if it is assumed only that R is residually small, then $R[S]$ is at minimum "close" to being residually small. Indeed, suppose that R is residually small, and consider an element $r_1 x^{s_1} + \cdots + r_k x^{s_k} \in R[S]$ such that $r^* := \sum_{i=1}^{k} r_i \neq 0$. Let I be an ideal of R not containing r^* such that $|R/I| < |R|$. As we saw from the proof of (1) of Theorem 1, the map $\varphi \colon R[S] \to R/I$ defined by $\varphi(a_1 x^{b_1} + \cdots + a_n x^{b_n}) := \overline{a}_1 + \cdots + \overline{a}_n$ is a ring surjection; let K be the kernel. It is clear that $r^* \notin K$. Moreover, $|R[S]/K| = |R/I| < |R| \leq |R[S]|$. We do not know if the residual smallness of R is sufficient for the residual smallness of $R[S]$ for all monoids S. If this is indeed the case, then we can modify (2) of Theorem 1 by removing the assumption that S is residually small and then replace (3) with "If S is a residually small monoid of size strictly larger than R, then $R[S]$ is residually small." As for (4) of Theorem 1, we cannot dispense with the assumption that $|S| \leq |R|$ as (2) of Corollary 2 witnesses.

Our next task is to study residual smallness in another class of rings we shall call *strong monoid rings*, defined as follows: let R be a ring and S be a monoid such that $|D(s)|$ is finite for every $s \in S$. Following the literature (cf. Henckell, Lazarus, & Rhodes [6]), S is said to be *finite $\mathcal{J}$-above*. Now let $R[[S]]$ denote the set of all functions $f\colon S \to R$ (the difference here is that we do not assume that the functions are finitely nonzero). Addition and multiplication is now defined exactly as for the monoid ring $R[S]$. The strong monoid rings over a ring R are, in general, very different objects than the monoid rings over R. We give an example to illustrate.

Example 1. *Let F be a finite field. Then $F[[\mathbb{N}]]$ has cardinality $2^{\aleph_0}$ whereas $|F[\mathbb{N}]| = \aleph_0$. Moreover, there is no monoid S such that $F[[\mathbb{N}]] \cong F[S]$.*

Proof. The additive monoid $\mathbb{N}$ is obviously finite $\mathcal{J}$-above, and $|F[[\mathbb{N}]]| = |F|^{\aleph_0} = 2^{\aleph_0}$. However, $F[\mathbb{N}] \cong F[X]$, and is thus countable.

As for the second assertion, suppose by way of contradiction that there is a monoid S such that $F[[\mathbb{N}]] \cong F[S]$. Observe that $F[[\mathbb{N}]] = F[[X]]$, the ring of formal power series in X over F. It is well-known that $F[[X]]$ is a discrete valuation ring, thus Noetherian. It follows that $F[S]$ is also Noetherian. Lemma 2 implies that S is finitely generated. But then $F[S]$ is countable, so cannot be isomorphic to $F[[\mathbb{N}]]$. This contradiction completes the proof. □

Our next theorem collects some facts about residually small strong monoid rings.

Theorem 2. *Let R be a ring and S be a monoid which is finite $\mathcal{J}$-above. Then the following hold:*

(1) *If R is residually small, then $R[[S]]$ is residually small.*
(2) *If $R[[S]]$ is infinite of cardinality greater than that of R, then $R[[S]]$ is residually small.*
(3) *If $R[[S]]$ is residually small and $|R[[S]]| = |R|$, then R is residually small.*

Proof. Assume that R is a ring and S is finite $\mathcal{J}$-above.

(1) Suppose that R is residually small, and let $f \in R[[S]]\backslash\{0\}$ be arbitrary. Then there is some $s^* \in S$ such that $f(s^*) \neq 0$. Set $f(s^*) := r^*$. Because R is residually small, there is an ideal I of R not containing r^* such that $|R/I| < |R|$. Next, set $\mathfrak{I} := \{f \in R[[S]]\colon f(s) \in I \text{ for every } s|s^*\}$. It is a simple task to verify that $\mathfrak{I}$ is an ideal of $R[[S]]$. Moreover, $f \notin \mathfrak{I}$. Now, let $\{s_1, \ldots, s_n\}$ be the set of divisors of s^*. Define $\varphi\colon (R/I)^n \to R[[S]]/\mathfrak{I}$ by $\varphi(\overline{r}_1, \ldots, \overline{r}_n) := r_1x^{s_1} + \cdots + r_nx^{s_n} \bmod(\mathfrak{I})$. Then it is easy to see that φ is a well-defined surjection. Hence we have

$$|R[[S]]/\mathfrak{I}| \leq |(R/I)^n| < |R| \leq |R[[S]]|.$$

This proves that $R[[S]]$ is residually small.

(2) Next, assume that $R[[S]]$ is infinite and of cardinality larger than that of R. Again, we let $f \in R[[S]]\backslash\{0\}$ be arbitrary. Pick $s^* \in S$ such that $f(s^*) \neq 0$, and set $\mathfrak{I} := \{f \in R[[S]]\colon f(s) = 0 \text{ for every } s|s^*\}$. As in the proof of (1), it is clear that $\mathfrak{I}$ is an ideal of $R[[S]]$. As above, let $\{s_1, \ldots, s_n\}$ be

the set of divisors of s^*, and let $\varphi\colon R^n \to R[[S]]/\mathfrak{I}$ be defined by $\varphi(r_1, \dots, r_n) := r_1x^{s_1} + \dots + r_nx^{s_n}$ mod$(\mathfrak{I})$. Then φ is surjective, and we have

$$|R[[S]]/\mathfrak{I}| \leq |R^n| < |R[[S]]|.$$

(3) Immediate from Lemma 1. □

Corollary 3. *Let R be a residually small ring.*

(1) *If S is a finite monoid, then $R[[S]]$ is residually small.*
(2) *The power series ring $R[[X_i\colon i < \rho]]$ in $\rho > 0$ indeterminates is residually small.*

Proof. Let R be as stated.

(1) Clear from (1) of Theorem 2 since every finite monoid is trivially finite $\mathcal{J}$-above.
(2) Simply note that $\bigoplus_\rho \mathbb{N}$ is finite $\mathcal{J}$-above and $R[[X_i\colon i < \rho]] \cong R[[\bigoplus_\rho \mathbb{N}]]$. □

To conclude this section, we present examples showing that the hypotheses of (1)–(3) of Theorem 2 cannot be dropped.

Example 2. *We cannot dispense with the assumption that R is residually small in (1) of Theorem 2. For instance, consider the ring $\mathbb{R}[[X]] = \mathbb{R}[[\mathbb{N}]]$. Note that $\mathbb{R}[[X]]$ has cardinality $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Moreover, for any proper ideal I of $\mathbb{R}[[X]]$, there is an injection of $\mathbb{R}$ into $\mathbb{R}[[X]]/I$. We conclude that $|\mathbb{R}[[X]]/I| = |\mathbb{R}[[X]]|$ for every proper ideal I of $\mathbb{R}[[X]]$, and thus $\mathbb{R}[[X]]$ is not residually small. This example also shows that we cannot remove the assumption that $|R[[S]]| > |R|$ in (2) of Theorem 2.*

Example 3. *We cannot remove the hypothesis that $|R[[S]]| = |R|$ in (3) of Theorem 2. For example, let κ be an infinite cardinal for which $\kappa < \kappa^{\aleph_0}$ (for instance, $\kappa = \aleph_{a+\omega}$ for some ordinal a)*[1] *and let F be a field of cardinality κ. Then $|F[[X]]| = \kappa^{\aleph_0}$, and $|F[[X]]/\langle X\rangle| = \kappa$. Thus $F[[X]]$ is residually small by Lemma 3. But F is a field, hence is not residually small.*

3. Residually small modules

We start with a definition that canonically generalizes the notion of residual smallness to modules.

Definition 2. *Let R be a ring and let M be an infinite R-module. Say that M is a residually small R-module if for every nonzero $m \in M$, there exists an R-submodule N_m of M such that $m \notin N_m$ and $|M/N_m| < |M|$.*

Our first goal is to show that the class of residually small modules generalizes another class of previously studied modules. To wit, an infinite module M over a ring R is called *homomorphically smaller* (HS for short) if $|M/N| < |M|$ for every nonzero submodule N of M (Oman & Salminen [18]). The existence of an abundance of residually small modules which are not HS follows immediately from Corollary 1. Conversely, not every HS module is residually small. To see this, let R be a ring. Note that every infinite simple R-module M (if any exist; they certainly do exist if R is an infinite field) is HS but not residually small. We shall prove that these are the only examples of

[1] This follows immediately from König's Theorem; cf. Jech [9], Corollary 5.14.

HS modules which are not residually small. To do this, we shall require the following results from the literature.

Lemma 5 ([18], (iv) of Proposition 3.2)**.** *If M is an HS module over the ring R, then $Ann(M) := \{r \in R : rM = \{0\}\}$ is a prime ideal of R.*

Lemma 6 ([18], Theorem 3.3 (partial))**.** *Let D be a domain with quotient field K and suppose that M is a faithful HS module over D. Then D is HS as a module over itself, and $D \subseteq M \subseteq K$ (up to D-module isomorphism).*

Finally we recall that a nonzero R-submodule N of an R-module M is *the minimum submodule of M* if $N \subseteq K$ for every nonzero R-submodule K of M (if such an N exists, it is easy to see that it is unique). We are now equipped to prove the first theorem of this section.

Theorem 3. *Suppose that M is a non-simple HS module over the ring R. Then M is also a residually small R-module.*

Proof. Let M be an HS module over the ring R which is not simple. By modding out the annihilator of M, by Lemma 5, there is no loss of generality in assuming that R is a domain. Since M is not simple, we deduce from Lemma 6 that

$$R \text{ is not a field.} \tag{3.1}$$

Suppose by way of contradiction that M is not residually small. Then there exists some nonzero $m \in M$ such that for all R-submodules N of M not containing m, $|M/N| = |M|$. Since M is HS, we deduce that every nonzero submodule of M contains m. Hence the cyclic R-module Rm is the minimum R-submodule of M. Invoking Lemma 6, we may assume that R is an R-submodule of M. But then Rm is the minimum R-submodule of R; in other words, Rm is the minimum ideal of R. But recall from (3.1) that R is a domain which is not a field, hence has no minimum ideals. This contradiction concludes the proof. ☐

Next, we show that the property of being residually small "passes down" easily between modules of the same cardinality. The next result generalizes Proposition 6 of [19].

Proposition 1. *Let $R \subseteq S$ be an extension of rings. Let M be an R-module and N be an S-module. Further, assume that M is an R-submodule of N and that $|M| = |N|$. If N is a residually small S-module, then M is a residually small R-module.*

Proof. Let R, S, M, and N be as stated and assume that N is a residually small S-module. Now let $m \in M\backslash\{0\}$ be arbitrary. Since $m \in N$ and N is a residually small S-module, there exists an S-submodule N' of N not containing m such that $|N/N'| < |N|$. Then of course $m \notin M \cap N'$. Moreover, the map $\varphi\colon M/(M \cap N') \to N/N'$ defined by $\varphi((M \cap N') + x) := N' + x$ is easily seen to be a well-defined injection. Thus $|M/(M \cap N')| \leq |N/N'| < |N| = |M|$. The proof is complete. ☐

As a simple consequence of the previous proposition, it is easy to see that residual smallness does not "pass upward" between modules of the same cardinality as easily.

Example 4. *Let R be a ring and suppose that M is a residually small R-module. If N is an R-module of the same cardinality as M which is not residually small, then $M \oplus N$ is not residually small.*

Despite the previous example, we have found a sufficient condition for residual smallness to transfer between certain module extensions. First, we recall a definition. Following terminology introduced in Oman [16], an R-submodule N of an infinite R-module M *large* if $|M/N| < |M|$. We can now reformulate the definition of a residually small module in terms of large submodules as follows:

Lemma 7. *Let M be an infinite R-module. Then M is residually small if and only if the intersection of the large R-submodules of M is trivial.*

Proof. M is residually small if and only if for every nonzero $m \in M$, there is a large submodule N of M not containing m if and only if for every nonzero $m \in M$, m is not in the intersection of the large R-submodules of M if and only if the intersection of the large R-submodules of M is trivial. □

Before stating our next proposition, we pause to present a simply corollary of this lemma.

Corollary 4. *Let R be a ring and suppose that M is a residually small R-module. Then M possesses infinitely many large R-submodules.*

Proof. Suppose not, and let $\{N_1, \ldots, N_k\}$ be the collection of large R-submodules of M. By the previous lemma, there is an injection of M into $M/N_1 \times \cdots \times M/N_k$. But elementary cardinal arithmetic shows that $|M/N_1 \times \cdots \times M/N_k| < |M|$, and this is a contradiction. □

Proposition 2. *Let M be an infinite R-module and suppose that N is a large R-submodule of M. Then M is residually small if and only if N is residually small.*

Proof. First, since the cosets of N in M all have cardinality $|N|$ and partition M, it follows that $|M| = |N| \cdot |M/N|$. Because N is large and M is infinite, basic set theory implies that $|N| = |M|$. Invoking Proposition 1, we see that N is residually small if M is so.

Conversely, suppose that N is residually small, and let K be a large R-submodule of N. As above, we have $|N/K| \cdot |(M/K)/(N/K)| = |M/K|$. But $(M/K)/(N/K) \cong_R M/N$. Thus (this is well-known, of course) $|M/N| \cdot |N/K| = |M/K|$. Now, $|M/N| < |M|$ and $|N/K| < |M|$. Since M is infinite, it follows that $|M/N| \cdot |N/K| = |M/K| < |M|$. Therefore, K is a large R-submodule of M. Recall that N is residually small; thus by Lemma 7, the intersection of the large R-submodules of N is trivial. As every large R-submodule of N is a large R-submodule of M, we conclude that the intersection of the large R-submodules of M is trivial as well. Lemma 7 yields that M is residually small. □

Our next task is to investigate how well the residually small property behaves with respect to direct sums and products. As with rings, the property transfers nicely.

Theorem 4. *Let R be a ring, ρ a nonzero cardinal, and let $\{M_i : i < \rho\}$ be a collection of R-modules. Then $M := \prod_{i<\rho} M_i$ (respectively, $M' := \bigoplus_{i<\rho} M_i$) is residually small if and only if M*

(respectively, M') is infinite and M_i is residually small for all i such that $|M_i| = |M|$ (respectively, for all i such that $|M_i| = |M'|$).

Proof. Assume that M and M' are defined as above. Suppose first that M (respectively, M') is residually small. Then by definition, M (respectively, M') is infinite. Now, if $i < \rho$ is such that $|M_i| = |M|$ (respectively, $|M_i| = |M'|$), then since there is an embedding of M_i into M (respectively, M'), we may apply Proposition 1 to conclude that M_i is residually small.

We now prove the converse. We only present the proof for the direct product; the proof for the direct sum is analogous. Thus suppose that M is infinite and that M_i is residually small for all i such that $|M_i| = |M|$ (note that there may be no such i). Let $(m_i) \in M$ be nonzero. Without loss of generality, we may assume that $m_0 \neq 0$. We now consider two cases.

Case 1: $|M_0| < |M|$. Then let $N := \{0\} \times \prod_{0<i<\rho} M_i$. Then of course $(m_i) \notin N$, and $|M/N| = |M_0| < |M|$.

Case 2: $|M_0| = |M|$. Then M_0 is residually small. Hence there exists an R-submodule N_0 of M_0 not containing m_0 such that $|M_0/N_0| < |M_0|$. Set $N := N_0 \times \prod_{0<i<\rho} M_i$. Then $(m_i) \notin N$ and $|M/N| = |M_0/N_0| < |M_0| = |M|$. This completes the proof. □

Corollary 5. *Let R be an infinite ring and suppose κ is a cardinal such that $\aleph_0 \leq \kappa \leq |R|$. Then R admits a residually small module of size κ if and only if R has a proper ideal of index less than κ.*[2]

Proof. Suppose first that M is a residually small R-module such that $|M| = \kappa$. Then there exists a proper submodule N of M such that $|M/N| < \kappa$. Pick any $m \in M \backslash N$. If I is the annihilator of $N + m$, then I is a proper ideal of R. Moreover, $R/I \cong_R R(N + m)$. Therefore, we have $0 < |R/I| = |R(N + m)| \leq |M/N| < \kappa$. Conversely, suppose that I is a proper ideal of R of index less than κ. Set $M := \bigoplus_\kappa R/I$. Then $|M| = \kappa$ and Theorem 4 implies that M is residually small. □

We are now equipped to have a brief discussion of projective modules over a residually small ring. It is not hard to find an example of a residually small ring R with an infinite projective module M which is not residually small:

Example 5. *Let R be the ring $\mathbb{Q} \times (\mathbb{Q}[[X]])$ and let I be the ideal $\mathbb{Q} \times \{0\}$. Then R is a residually small ring and I is an infinite projective R-module which is not residually small.*

Proof. It is immediate from (2) of Theorem 2 that $\mathbb{Q}[[X]]$ is a residually small ring. Theorem 4 of [19] implies that R is also a residually small ring. Now, I is a simple R-module, hence is not residually small. Since $I \oplus (\{0\} \times (\mathbb{Q}[[X]])) = R$, we see that I is projective. □

Notice that the R-module I in the above example has smaller cardinality than R. We will prove that if M is a projective module over a residually small ring R whose cardinality is at least that of R, then M is a residually small R-module. In fact, we can prove even more.

[2] Recall that the *index* of an ideal I in a ring R is the cardinality of the ring R/I.

Proposition 3. *Let R be a residually small ring and let M be an R-module of size at least $|R|$. If M is a submodule of a free R-module, then M is residually small.*

Proof. Assume that R is a residually small ring and M is an R-module such that $|R| \leq |M|$ and $M \leq_R F$ for some free R-module F. Then there is a nonempty set I such that $F := F_I = \{f\colon I \to R\colon f \text{ is finitely nonzero}\}$. For $f \in M$, recall that the support of f is defined by $\text{supp}(f) := \{i \in I\colon f(i) \neq 0\}$. Now set

$$X := \bigcup_{f \in M} \text{supp}(f).$$

Clearly, M embeds into F_X. Therefore,

$$|M| \leq |F_X|. \tag{3.2}$$

Since each $f \in F$ is finitely nonzero, it follows that

$$|X| \leq \aleph_0 \cdot |M| = |M|. \tag{3.3}$$

By assumption, $|R| \leq |M|$. Invoking (3.3), we see that

$$|F_X| = \max(|X|, |R|) \leq |M|. \tag{3.4}$$

We deduce from (3.2) and (3.4) that $|M| = |F_X|$. Recall that R is a residually small ring, hence also a residually small R-module, by definition. Applying Theorem 4, F_X is a residually small R-module. Finally, we invoke Proposition 1 to conclude that M is residually small. $\square$

We shall shortly invoke the theory of injective modules to conclude our study. First, let us agree to call an R-module M *R-big* if M is infinite and $|M| > |R|$. Our first proposition on R-big modules showcases the utility of Proposition 1. First, we prove a lemma.

Lemma 8. *Let V be a vector space over a field F. Then V is residually small if and only if $\dim_F(V)$ is infinite and larger than $|F|$.*

Proof. Suppose first that V is residually small. Assume by way of contradiction that $\dim_F(V)$ is finite or $\dim_F(V) \leq |F|$. First, suppose $\dim_F(V)$ is finite. Since V is infinite (by definition of 'residually small'), it follows that F is an infinite field. Basic cardinal arithmetic implies that $|V| = |F|$. But then $|V/W| = |F|$ for every proper subspace W of V (since V/W contains an isomorphic copy of F), and this contradicts the fact that V is residually small. Now assume that $\dim_F(V) \leq |F|$. As above, we deduce that F is infinite. But then $|V| = \max(|F|, \dim_F(V)) = |F|$, and we have a contradiction as above.

Conversely, assume that $\dim_F(V)$ is infinite and larger than $|F|$. Set $\kappa := \dim_F(V)$. Then $V \cong_F \bigoplus_\kappa F$ (the direct sum of κ copies of F). Let $\mathbf{v} \in V$ be nonzero. Then we may assume without loss of generality that $\mathbf{v} = (f_0, f_1, \ldots, f_n, 0, 0, 0, \ldots)$, where $n \in \mathbb{N}$ and each f_i is a nonzero element of F. Observe that $\mathbf{v} \notin W := \{0\} \oplus (\bigoplus_{0<i<\kappa} F)$. Moreover, it is clear that $V/W \cong_F F$. Therefore, $|V/W| = |F| < \dim_F(V) = |V|$, and V is residually small. $\square$

We now apply the previous lemma and Proposition 1 to obtain the following result.

Proposition 4. *Let D be a domain and let M be a torsion-free D-big module. Then M is residually small.*

Proof. Let D and M be as stated. Now let $S := D\backslash\{0\}$. Then S is a multiplicative subset of D. Moreover, the module $S^{-1}M$ is torsion-free over the fraction field $F := D_S$. Since $|M| > |D|$, we have $|S^{-1}M| = |M| > |D| = |F|$. We deduce that $\dim_F(S^{-1}M)$ is infinite and larger than $|F|$. By Lemma 8, we see that $S^{-1}M$ is a residually small F-vector space. Invoking Proposition 1, it follows that M is a residually small D-module. $\square$

For further results on big modules, we shall require another lemma.

Lemma 9. *Let R be a ring and let M be an infinite R-module.*

(1) *If M possesses a minimum R-submodule, then M is not residually small.*
(2) *If M is not residually small, then there exists an R-submodule N of M such that $|M/N| = |M|$ and M/N has a minimum R-submodule.*

Proof. Let R be a ring and M an infinite R-module.

(1) Suppose that N is a minimum R-submodule of M. Then N is cyclic; let $N = Rn$. Thus if K is an R-submodule of M not containing n, then $K = \{0\}$. It follows that M is not residually small.

(2) Assume that M is not residually small. Then there is some nonzero $m \in M$ such that

$$\text{for all } R\text{-submodules } K \text{ of } M \text{ not containing } m, |M/K| = |M|. \tag{3.5}$$

By Zorn's Lemma, there exists an R-submodule N of M which is maximal with respect to not containing m. Thus $|M/N| = |M|$ by (3.5). We claim that $(N + Rm)/N$ is the minimum R-submodule of M/N. First, since $m \notin N$, we see that $(N + Rm)/N$ is nontrivial. Next, consider a nontrivial R-submodule X of M/N. Then $X = K/N$ for some R-submodule K of M properly containing N. By maximality of N, we deduce that $m \in K$. But then $N + Rm \subseteq K$, and we conclude that $(N + Rm)/N \subseteq K/N$. $\square$

Remark 1. *If M is an infinite R-module which is not residually small, then M need not possess a minimum submodule. As we have seen, the domain $\mathbb{R}[[X]]$ is not a residually small ring. Since $\mathbb{R}[[X]]$ is a domain which is not a field, it follows that $\mathbb{R}[[X]]$ does not have a minimum ideal.*

We now recall a standard fact on injective modules (as usual, $E(M)$ denotes the injective envelope of an R-module M).

Fact 1 (Lam [10], (1) - (4) of Theorem 3.52)**.** *Let M be an injective module over a ring R (not assumed to be commutative). The following are equivalent:*

(1) *M is indecomposable.*
(2) *M is nontrivial and $M = E(M')$ for any nonzero R-submodule $M' \subseteq M$.*
(3) *M is uniform.*
(4) *$M = E(U)$ for some uniform R-module U.*

We are almost equipped to show that every infinite module over a finite ring is residually small. We shall require the following lemma. The assertions of this lemma are surely in the literature, but since the proof is short, we include it.

Lemma 10. *Let R be a ring and let M be an R-module. Further, suppose M possesses a minimum R-submodule N. Then the following hold:*

(1) *N is simple.*
(2) *N is the minimum R-submodule of $E(M)$.*
(3) *$E(M) = E(N)$.*

Proof. We assume that N is the minimum R-submodule of the R-module M.
(1) Trivial.
(2) Immediate, since $E(M)$ is an essential extension of M.
(3) By (2), $E(M)$ is uniform. We now invoke Fact 1 to complete the proof. □

Proposition 5. *Let R be a finite ring. Every infinite R-module is residually small.*

Proof. Let R be a finite ring. By way of contradiction, suppose there is an infinite R-module M which is not residually small. By Lemma 9, we may assume that M has a minimum R-submodule. By (2) of Lemma 10, $E(M)$ is uniform. Thus $E(M)$ is indecomposable by Fact 1. Being finite, R is certainly Artinian. It is well-known that all indecomposable injective modules over an Artinian ring are finitely generated (cf. Matlis [14], (2) of Theorem 3.11). Thus $E(M)$ is an infinite R-module that is finitely generated over the finite ring R. This is absurd, and the proof is concluded. □

In fact, we can strengthen the statement of Proposition 5. Our next corollary was proven by Varadarajan ([24]) for any finite ring, whether commutative or not. We give a shorter proof in the commutative case.

Corollary 6 ([24], Theorem 2.1)**.** *Let R be a finite ring. Every infinite R-module is residually finite.*

Proof. Let R be a finite ring and let M be an infinite R-module. Choose $m \in M \backslash \{0\}$ arbitrarily. Now let N be a submodule of M maximal with respect to not containing m. We claim that M/N is finite. Suppose not. As we have seen, $(N + Rm)/N$ is the minimum submodule of M/N. Therefore, (1) of Lemma 9 implies that M/N is not residually small. This contradicts Proposition 5, and the proof is complete. □

A question now arises: if R is a ring for which every infinite R-module is residually small, must R be finite? Suppose that R is any such ring. By the argument given in the previous corollary, it follows that every infinite R-module is residually finite (using Varadarajan's terminology, we say that R is an *RF ring*). It is known that there exist commutative RF rings of every infinite cardinality. For example, every infinite Boolean ring is an RF ring (see [24], p. 348). Thus the answer to the above query is a resounding "no."

Next on our agenda is an investigation of the residual smallness of R-big modules for infinite rings R. We give the following necessary and sufficient condition on an infinite ring R for all R-big modules to be residually small.

Proposition 6. *Let R be an infinite ring. Then every R-big module is residually small if and only if $|E(M)| \leq |R|$ for every simple R-module M.*

Proof. Simply observe that there exists an R-big module which is not residually small if and only if (by Lemma 9) there exists an R-big module with a minimum submodule if and only if (by Lemma 10) there exists an R-big injective module with a minimum submodule if and only if (again, by Lemma 10) $|E(M)| > |R|$ for some simple R-module M. □

The previous proposition immediately raises the following question: can the rings R such that $|E(M)| \leq |R|$ for every simple R-module M be classified? Is there even an infinite ring R and a simple R-module M such that $|E(M)| > |R|$? We do not know the answers to these questions. However, there are large classes of infinite rings R for which $|E(M)| \leq |R|$ for every simple R-module M. For example, Noetherian rings have this property. The following is well-known:

Fact 2 (Eklof [3], Remark 1)**.** *Let R be a ring. Then R is Noetherian if and only if $|E(M)| = |M|$ for every infinite R-module M with $|M| \geq |R|$.*

Using this result, we can easily show that all R-big modules are residually small if R is Noetherian.

Proposition 7. *Let R be a Noetherian ring. All R-big modules are residually small.*

Proof. By Proposition 5, we may assume that R is infinite. Let M be a simple R-module. By Proposition 6, it suffices to prove that $|E(M)| \leq |R|$. Suppose $|E(M)| > |R|$. Now choose an R-submodule N of $E(M)$ such that $M \subseteq N \subseteq E(M)$ and $|N| = |R|$. Since $E(N) \cong E(M)$, also $|E(N)| > |R| = |N|$. However, this contradicts Fact 2. □

It is easy to find Noetherian rings R for which R possesses residually small modules of cardinality less than $|R|$, residually small modules of size $|R|$, and residually small modules of cardinality greater than $|R|$. For instance, let $R := \mathbb{Z}[[X]]$. Then R is Noetherian, and hence every R-big module is residually small. Moreover, (2) of Theorem 2 implies that R is residually small as a module over itself. Lastly, $R/\langle X\rangle \cong_R \mathbb{Z}$ is a residually small R module of smaller cardinality than that of R. A natural question is what happens if one strengthens the hypothesis by replacing "Noetherian" with "Artinian." Since every Artinian ring is a finite product of Artinian local rings, we consider only the local case. We shall require the following result:

Lemma 11 ([16], Proposition 8)**.** *Let R be an infinite Artinian local ring. Then $|I| = |R|$ for every nonzero ideal I of R and $|R/I| = |R|$ for every proper ideal I of R.*

Proposition 8. *Let R be an Artinian local ring and let M be an R-module. Then M is residually small if and only if M is R-big.*

Proof. Assume R is as stated and that M is an R-module. If M is R-big, then Proposition 7 implies that M is residually small. Conversely, suppose that M is residually small. If $|M| \leq |R|$, then by Corollary 5, R has a proper ideal of index less than $|R|$. However, this is precluded by Lemma 11. □

While we don't know, in general, if big modules are always residually small, we can show that every "sufficiently large" module is residually small. Toward this end, we introduce additional terminology.

Let M be a module over a ring R and let $I \subseteq M$. Then we say that I is an *independent* subset of M if I generates a direct sum. In other words, I is independent if for all distinct $m_1, \ldots, m_n \in I$: if $r_1 m_1 + \cdots + r_n m_n = 0$ for some $r_1, \ldots, r_n \in R$, then each $r_i m_i = 0$. It follows easily from Zorn's Lemma that M possesses a maximal independent subset. Moreover, Andreas Ecker has shown the following:

Lemma 12 ([17], Proposition 4)**.** *Let R be an infinite ring and let I be a maximal independent set in an R-module M. Then the following hold:*

(1) *If $I = \varnothing$, then $M = \{0\}$.*
(2) *If $|I| = 1$, then $|M| \leq 2^{|R|}$.*
(3) *If $I > 1$, then $|M| \leq |I|^{|R|}$.*

We now prove that all sufficiently large R-modules are residually small.

Proposition 9. *Let R be a ring and let M be an infinite R-module such that $2^{|R|} < |M|$. Then M is residually small.*

Proof. If R is finite, then we are done by Proposition 5. Thus assume that R is infinite and that $2^{|R|} < |M|$. Suppose by way of contradiction that M is not residually small. By Lemma 9, there exists an R-module K of the same cardinality as M such that K has a minimum submodule N. Let I be a maximal independent subset of K. The minimality of N implies that $|I| = 1$. An application of Lemma 12 shows that $|M| = |K| \leq 2^{|R|}$, and we have reached a contradiction. □

We conclude this section by proving that it is consistent with ZFC that for many rings R which are not Noetherian, all R-big modules M are residually small. Toward this end, recall that *the Generalized Continuum Hypothesis* (GCH) is the assertion that for every infinite cardinal κ, there is no cardinal strictly between κ and 2^κ. It is well-known (from Gödel and Cohen's work) that GCH can neither be proved nor refuted in ZFC.

We now recall another result from [3]. For a ring R, let $\mu(R) := \mu$ be the least cardinal number μ such that every ideal of R can be generated by fewer than μ elements. Then the following holds:

Lemma 13 ([3], Theorem 1)**.** *Let R be a ring and let κ be an infinite cardinal such that $\kappa \geq |R|$. Further, let M be an R-module of cardinality κ. If $\sum_{\alpha<\mu(R)} \kappa^\alpha = \kappa$, then $|E(M)| = |M|$.*

We conclude this section with a proposition (we refer the reader to [2] for additional results concerning bounding the cardinality of generating sets of ideals).

Proposition 10. *Assume GCH and let κ be a regular cardinal. Further, let R be a ring of cardinality κ with the property that every ideal of R can be generated by fewer than $|R|$ elements. Then every R-big module is residually small.*

Proof. By Proposition 6, the proof of Proposition 7, and Lemma 13, it suffices simply to prove that $\sum_{\alpha<\mu(R)} \kappa^\alpha = \kappa$. By the condition on R, we have $\mu(R) \leq |R| = \kappa$. Since κ is a regular cardinal and GCH holds, it follows that $\kappa^\beta = \kappa$ for every nonzero cardinal $\beta < \kappa$ (cf. Theorem 5.15 of [9]). Therefore,

$$\sum_{\alpha<\mu(R)} \kappa^\alpha = \mu(R) \cdot \kappa = \kappa.$$

The proof is complete. $\square$

4. Open Questions

We conclude the paper with the following open questions for further study.

Open Problem 1. *Suppose that R is a residually small ring and S is a monoid. Must $R[S]$ be residually small?*

Open Problem 2. *Let R be a ring. Are all R-big modules residually small?*

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