POLYNOMIAL AND POWER SERIES RINGS WITH FINITE QUOTIENTS

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Abstract. We determine the rings R with the property that the quotient ring $R[X]/I$ (respectively, $R[[X]]/I$) is finite for every nonzero ideal I of the polynomial ring $R[X]$ (respectively, of the power series ring $R[[X]]$). We also classify the rings R such that $R[X]/I$ ($R[[X]]/I$) is a finite left $R[X]$-module ($R[[X]]$-module) for every nonzero left ideal I of $R[X]$ (of $R[[X]]$).

1. Introduction

Suppose that F is a finite field, and fix some nonzero polynomial $f(X) \in F[X]$ of degree n. As is well-known, the polynomial ring $F[X]$ is a Euclidean domain. Thus via the Division Algorithm, every member of the quotient ring $F[X]/\langle f(X)\rangle$ can be expressed in the form $\langle f(X)\rangle + r(X)$, where $r(X) \in F[X]$ is zero or of degree less than n. It follows that $|F[X]/\langle f(X)\rangle| \leq |F|^n$, and therefore $F[X]/\langle f(X)\rangle$ is finite. Since $F[X]$ is a principal ideal domain, we conclude that $F[X]/I$ is finite for every nonzero ideal I of $F[X]$. Consider now the ring $F[[X]]$ of formal power series in the variable X with coefficients in F. It is also well-known that $F[[X]]$ is a discrete valuation ring, and $\{\langle X^n\rangle \colon n \geq 0\}$ is the collection of nonzero ideals of $F[[X]]$. Moreover, it is easy to see that for any non-negative integer n, $|F[[X]]/\langle X^n\rangle| = |F|^n$. Therefore, $F[[X]]$ too has the property that $F[[X]]/I$ is finite for every nonzero ideal I of $F[[X]]$. The purpose of this paper is to investigate a sort of converse. In particular, we classify the rings R with the property that $R[X]/I$ (respectively, $R[[X]]/I$) is finite for every nonzero ideal I of $R[X]$ (of $R[[X]]$). We then consider an analogous problem involving one-sided ideals.

There is considerable literature related to the problem we investigate in this note. In [2], Chew and Lawn define a ring R to be *residually finite* provided R/I is finite for every nonzero ideal I of R. Levitz and Mott ([9]) extend Chew and Lawn's results to rings without identity and say such rings have the *finite norm property.* This concept was also considered by Ion, Militaru, and Niţă in [6] and [7]. In the latter two papers, the authors call an infinite, non-simple unital ring R a *finite quotient ring* if R/I is finite for every nonzero ideal I of R. Using Chew and Lawn's terminology, a finite quotient ring is an infinite, non-simple, residually finite ring. Unfortunately, the phrase "residually finite ring" is multiply defined in the literature. For example, Lewin ([10]) along with Orzech and Ribes ([12]) defines a ring R to be residually finite if for every $r \in R\backslash\{0\}$, there exists an ideal I_r of R such that $r \notin I_r$ and R/I_r is finite. Similarly, Varadarajan ([13]–[14]) calls a left (right) module M over a ring R a *residually finite module* if for every nonzero $m \in M$, there is a submodule N_m of M not containing m such that M/N_m is finite. He also defines a ring R to be an

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RRF ring (LRF ring) if every right R-module (left R-module) is residually finite. To make matters even more confusing (given the terminology introduced earlier by Lewin, Orzech, and Ribes), he calls a ring an RF ring if it is simultaneously an RRF ring and an LRF ring. Faith simplified and generalized some of Varadarajan's results in [3], retaining Varadarajan's definitions of RRF ring and LRF ring.

To avoid exacerbating the terminological issues outlined above, we settle on nomenclature introduced in [11] which, as far as we know, does not conflict with the literature. Call a ring R (not assumed to have an identity) *homomorphically finite* (abbreviated HF) if R/I is finite for every nonzero ideal I of R. Similarly, say that a left R-module M is HF provided M/N is finite for every nonzero R-submodule N of M.

In this paper, we classify the rings R for which the polynomial and power series rings $R[X]$ and $R[[X]]$, respectively, are HF. We also prove an analogous one-sided result characterizing the finite fields. We conclude the introduction by mentioning that *throughout the paper, when we state that R is a ring, we assume only that R is associative. We do not assume commutativity nor the existence of a multiplicative identity.*

2. Two-sided results

We begin by recalling some terminology. Recall that if R is a ring, then an (left, right, or two-sided) ideal I of R is *nilpotent* if there is a positive integer n such that $I^n = \{0\}$; R is a *nilpotent ring* if R is a nilpotent ideal. A ring R is *left Artinian* if there is no infinite, strictly decreasing sequence of left ideals of R with respect to set inclusion. We shall make use of the following result from the literature.

Fact 1 ([8], p. 22). *If R is a left Artinian ring with no nonzero nilpotent left ideals, then R is a semisimple ring with identity.*

We use this fact to prove our first lemma.

Lemma 1. *Suppose R is a finite simple ring which is not nilpotent. Then R has an identity.*

Proof. Let R be a finite (hence left Artinian) simple ring which is not nilpotent. By Fact 1, it suffices to prove that R has no nonzero nilpotent left ideals. Suppose by way of contradiction that R has a nonzero nilpotent left ideal I. Then $I^n = \{0\}$ for some positive integer n. Now, IR is a two-sided ideal of R. Moreover, $(IR)^n = I(RI)^{n-1}R \subseteq II^{n-1}R = I^nR = 0R = \{0\}$. Hence IR is a nilpotent ideal of R. Since R is not nilpotent, $IR \neq R$, and the simplicity of R implies that $IR = \{0\}$. Recall that $ann_R(R) := \{r \in R\colon rR = \{0\}\}$ is a two-sided ideal of R. Because $IR = \{0\}$, we see that $I \subseteq ann_R(R)$. But then $ann_R(R)$ is nontrivial. As R is simple, it follows that $ann_R(R) = R$. Therefore $R^2 = \{0\}$, contradicting that R is not nilpotent. □

Before proceeding, we shall require another lemma.

Lemma 2. *Let G be an abelian group, and suppose that H and K are subgroups of G such that $H \subseteq K$. Then $|G/K| \leq |G/H|$.*

Proof. Suppose that G, H, and K are as stated. Then the map $H + g \mapsto K + g$ is a well-defined surjection of G/H onto G/K. By the Axiom of Choice, there is an injection of G/K into G/H, and the result follows. □

We are now equipped to classify the rings R for which $R[X]$ is homomorphically finite.

Theorem 1. *Let R be a ring. Then $R[X]$ is homomorphically finite if and only if R is a finite simple ring with identity.*

Proof. Clearly we may assume that R is a nonzero ring. Suppose first that R is a finite simple ring with 1, and let I be a nonzero ideal of $R[X]$. Let $f(X) \in I$ be nonzero of degree n and let r_n be the leading coefficient of $f(X)$. Because R is simple, $Rr_nR = R$. Thus $1 \in Rr_nR$, and it follows that $Rf(X)R$ contains a *monic* polynomial $g(X)$ of degree n. Since $Rf(X)R \subseteq I$, we deduce that $g(X) \in I$. Now, if $n = 0$, then $I = R[X]$ and so $R[X]/I$ is certainly finite. Thus suppose that $n > 0$. Next, let $\langle g(X)\rangle$ be the two-sided ideal of $R[X]$ generated by $g(X)$ and M be the left R-submodule of $R[X]/\langle g(X)\rangle$ generated by $\overline{1}, \overline{X}, \ldots, \overline{X}^{n-1}$. Clearly $|M| \leq |R|^n$, and so M is finite. Moreover, since $g(X)$ is monic of degree n, we see that $\overline{X}^n \in M$. One shows easily by induction that $\overline{X}^m \in M$ for every integer $m \geq n$. It is now clear that $M = R[X]/\langle g(X)\rangle$, and hence $R[X]/\langle g(X)\rangle$ is finite. Invoking Lemma 2, we conclude that $R[X]/I$ is also finite, as claimed.

Conversely, assume that $R[X]$ is homomorphically finite. The map $\varphi\colon R[X] \to R$ defined by $\varphi(r_0 + r_1X + \cdots + r_nX^n) := r_0$ is a ring map from $R[X]$ onto R with nontrivial kernel. Because $R[X]$ is homomorphically finite, we see that R is a finite ring. Next we claim that R is simple. Suppose by way of contradiction that I is a proper, nonzero ideal of R. Then $I[X] := \{i_0 + i_1X + \cdots + i_nX^n : n \geq 0, i_k \in I\}$ is easily seen to be a nonzero ideal of $R[X]$. Moreover,

$$R[X]/I[X] \cong (R/I)[X]. \tag{2.1}$$

Since I is a proper ideal of R, R/I is a nonzero ring. Therefore, $(R/I)[X]$ is infinite. But then by (2.1), so too is $R[X]/I[X]$. This contradicts our assumption that $R[X]$ is homomorphically finite. Next, we prove that R is not nilpotent. It suffices to prove that $R^2 = R$. Suppose not. Then the simplicity of R implies that $R^2 = \{0\}$. Choose any nonzero $r \in R$. Then the cyclic group (r) is an ideal of $R[X]$. But $rX \notin (r)$, a contradiction. Applying Lemma 1, we conclude that R has an identity. This concludes the proof. □

Our next task is to classify the rings R for which the power series ring $R[[X]]$ is homomorphically finite. Toward this end, we establish another lemma. Compare to the first paragraph of the Introduction.

Lemma 3. *Let R be a ring with identity. Then R is simple if and only if every nonzero ideal of the power series ring $R[[X]]$ is of the form $\langle X^n\rangle$ for some $n \geq 0$.*

Proof. Let R be a ring with identity. Suppose first that R is simple, and let I be a nonzero ideal of $R[[X]]$. For any nonzero $j(X) := \sum_{i=0}^{\infty} j_iX^i \in I$, let $l(j(X))$ denote the least m such that $j_m \neq 0$. Now choose some nonzero $f(X) := \sum_{i=0}^{\infty} a_iX^i \in I$ for which $l(f(X)) := n$ is minimal. Because R

is simple, $Ra_nR = R$. Therefore, $Rf(X)R$ contains a power series $g(X) := \sum_{i=0}^{\infty} b_iX^i$ such that $b_n = 1$ and $b_j = 0$ for $0 \leq j < n$. Clearly $g(X) \in I$. Moreover, we can write

$$g(X) = (X^n) \cdot h(X), \text{ where } h(X) = 1 + b_{n+1}X + b_{n+2}X^2 + \cdots . \tag{2.2}$$

It is well-known that if R is a ring with identity, then a power series $k(X) \in R[[X]]$ is a unit of $R[[X]]$ if and only if the constant term of $k(X)$ is a unit of R (cf. [5], Proposition 5.9). We now deduce from (2.2) that $X^n \in I$. Moreover, it is immediate from the minimality of n that $I \subseteq \langle X^n \rangle$. We conclude that $I = \langle X^n \rangle$.

Conversely, suppose that $\{\langle X^n \rangle : n \geq 0\}$ is the collection of nonzero ideals of $R[[X]]$ and let I be a nonzero ideal of R. Then $I[[X]] := \{\sum_{j=0}^{\infty} i_jX^j : i_j \in I\}$ is a nonzero ideal of $R[[X]]$; thus $I[[X]] = \langle X^n \rangle$ for some $n \geq 0$. Let $r \in I$ be nonzero. Then $r \in I[[X]] = \langle X^n \rangle$. It is easy to see that $\langle X^n \rangle$ contains nonzero constants if and only if $n = 0$. Therefore, $I[[X]] = \langle X^0 \rangle = R[[X]]$, from which it follows that $I = R$. Thus R is simple, and the proof is concluded. □

We now classify the rings R for which $R[[X]]$ is HF.

Theorem 2. *Let R be a ring. Then $R[[X]]$ is homomorphically finite if and only if R is a finite simple ring with identity.*

Proof. Again, we may assume that R is a nonzero ring. If $R[[X]]$ is homomorphically finite, then a *mutatis mutandis* adaptation of the proof of Theorem 1 shows that R is a finite simple ring with identity.

Conversely, suppose that R is a finite simple ring with identity. By the previous lemma, it suffices to prove that $R[[X]]/\langle X^n \rangle$ is finite for every $n \geq 0$. This is patent for $n = 0$, and for $n > 0$, simply observe that the left R-submodule M of $R[[X]]/\langle X^n \rangle$ generated by $\{\overline{1}, \overline{X}, \ldots, \overline{X}^{n-1}\}$ exhausts $R[[X]]/\langle X^n \rangle$ and, moreover, that $|M| \leq |R|^n$. The finiteness of R now implies the finiteness of M, and the proof is concluded. □

The following corollary is immediate from our work above, the Artin-Wedderburn Theorem, and Wedderburn's Theorem on Finite Division Rings.

Corollary 1. *Let R be a ring. The following are equivalent:*

1. *$R[X]$ is homomorphically finite.*
2. *$R[[X]]$ is homomorphically finite.*
3. *R is a finite simple ring with identity.*
4. *$R \cong M_{n\times n}(F)$ for some positive integer n and finite field F.*

3. One-sided results

We now consider one-sided ideals. Recall from the Introduction that if R is a ring and M is a left R-module, then by definition M is homomorphically finite if and only if M/N is finite for every nonzero R-submodule N of M. We shall require the following lemma:

Lemma 4 ([4], p. 101)**.** *Let R be a nonzero ring for which R has no proper, nonzero left ideals. Then either R is a division ring or $R \cong \mathbb{Z}/\langle p \rangle$ with trivial (zero) multiplication for some prime p.*

Proposition 1. *Let R be a nonzero ring with nontrivial multiplication. Then R is a finite field if and only if there exists a ring extension S of R satisfying the following:*

(1) *R is a proper homomorphic image of S (as a ring),*
(2) *S is a homomorphically finite (left) S-module over itself, and*
(3) *S/I^e is an infinite left S-module for every proper nonzero left ideal I of R, where I^e is the left ideal of S generated by I.*

Proof. Assume that R is a nonzero ring with nontrivial multiplication. If R is a finite field, then we may take $S := R[X]$. Conversely, suppose S is a ring extension of R which satisfies (1)-(3) above. By (1), R is a proper homomorphic image of S (that is, there is a surjective ring map $\varphi\colon S \to R$ with nonzero kernel); (2) implies that R is finite. We conclude immediately from (2) and (3) that R has no proper nonzero left ideals. By Lemma 4, R is either a division ring or there is a prime p such that R is isomorphic to $\mathbb{Z}/\langle p\rangle$ with trivial multiplication. By assumption, the multiplication on R is nontrivial. Hence R is a finite division ring, and so a finite field. □

Theorem 3. *Let R be a nonzero ring. The following are equivalent:*

(1) *R is a finite field.*
(2) *$R[X]$ is homomorphically finite as a left module over itself.*
(3) *$R[[X]]$ is homomorphically finite as a left module over itself.*

Proof. Suppose that R is a nonzero ring.

(1)⇒(2): apply Theorem 1.

(1)⇒(3): by Theorem 2.

(2)⇒(1): suppose that $R[X]$ is homomorphically finite as a left module over itself. We claim that the multiplication on R is nontrivial. Suppose by way of contradiction that $r_1r_2 = 0$ for all $r_1, r_2 \in R$. Take $r \in R$ to be nonzero and consider the additive cyclic subgroup (r) of $R[X]$ generated by r. As in the proof of Theorem 1, the cyclic group (r) is an ideal of $R[X]$. But $Xr \notin (r)$, and we have reached a contradiction. Now set $S := R[X]$. One checks easily that (1)-(3) of Proposition 1 are satisfied, and thus R is a finite field.

(3)⇒(1): replace "$[X]$" with "$[[X]]$" in the proof of the previous implication. □

Remark 1. *Note trivially that in the statement of Theorem 3, we may replace the word "left" with "right" and the result still holds.*

4. An open question

We conclude this note with a brief discussion of the following:

Question 1. *Let R be a finite simple ring with identity. Can one classify the homomorphically finite rings S such that $R[X] \subseteq S \subseteq R[[X]]$?*

Even in the commutative case, this appears to be a difficult problem. In Theorem 4.2 of [11], it was shown that a commutative domain D with identity is homomorphically finite if and only if D is Noetherian of (Krull) dimension at most one with all residue fields finite. If we take $R := F$ to be a finite field, there are rings properly between $F[X]$ and $F[[X]]$ which are homomorphically finite and others which are not. To verify the former assertion, let S be a countable elementary subring[1] of $F[[X]]$ which contains $F[X]$. Then S is a DVR with maximal ideal $\langle X\rangle \cap S$. Moreover, $S/(\langle X\rangle \cap S)$ maps canonically into $F[[X]]/\langle X\rangle \cong F$; hence S has a finite residue field. Invoking Theorem 4.2 of [11], we deduce that S is HF. On the other hand, $F[[X]]$ has size $2^{\aleph_0}$ whereas $F[X]$ is countable. Thus there exists $\zeta \in F[[X]]$ which is transcendental over $F[X]$. But then $F[X]$ is a proper infinite homomorphic image of $F[X](\zeta)$, and we see that $F[X](\zeta)$ is not HF. We leave a deeper investigation of Question 1 to the interested reader.

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[1]That is, S is a subring of $F[[X]]$ and both S and $F[[X]]$ satisfy the same first-order sentences in the language of rings. The existence of such an S follows from the Lowenheim-Skolem Theorem; we refer the reader to [1] for further details.