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# MODULES WHICH ARE ISOMORPHIC TO THEIR FACTOR MODULES

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***Let R be commutative ring with identity and let M be an infinite unitary R-module. Call M homomorphically congruent (HC for short) provided $M/N \cong M$ for every submodule N of M for which $|M/N| = |M|$. In this article, we study HC modules over commutative rings. After a fairly comprehensive review of the literature, several natural examples are presented to motivate our study. We then prove some general results on HC modules, including HC module-theoretic characterizations of discrete valuation rings, almost Dedekind domains, and fields. We also provide a characterization of the HC modules over a Dedekind domain, extending Scott's classification over $\mathbb{Z}$ in [22]. Finally, we close with some open questions.***

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## 1. BACKGROUND

Here, all rings are commutative with identity and all modules are unitary left modules. In universal algebra, an *algebra* is a pair $(X, \mathbf{F})$ consisting of a set $X$ and a collection $\mathbf{F}$ of operations on X (there are no restrictions placed on the arity of these operations). In case $\mathbf{F}$ is countable and all operations have finite arity, then $(X, \mathbf{F})$ is called a *Jónsson algebra* provided every proper subalgebra of $X$ has smaller cardinality than $X$. Such algebras are of particular interest to set theorists, and many articles have been written on this topic; we refer the reader to [1] for an excellent survey of Jónsson algebras.

In the early 1980s, Robert Gilmer and William Heinzer translated these notions to the realm of commutative algebra. In [7], they define a module $M$ over a commutative ring $R$ with identity to be a *Jónsson module* provided every proper submodule of $M$ has smaller cardinality than $M$. They applied and extended their results in several subsequent papers ([4–6]). The first author continued this study in [12, 13], and [19].

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Much earlier, Scott studied a related algebraic structure. Specifically, he classified the abelian groups $G$ for which $H \cong G$ for every subgroup $H$ of $G$ of the same cardinality as $G$ ([22]). In [16], the first author extends Scott's result to infinite modules over a Dedekind domain; in [14] and [17], he studies this concept over more general classes of rings, calling an infinite module $M$ over a ring $R$ *congruent* if and only if every submodule $N$ of $M$ of the same cardinality as $M$ is isomorphic to $M$ (note that every Jónsson module is trivially congruent).

Variants of this notion have also received attention in model theory, group theory, and topology. In [2], Droste calls a structure $S$ *elementarily $\kappa$-homogeneous* provided every two substructures of cardinality $\kappa$ are elementarily equivalent (that is, every two substructures of cardinality $\kappa$ satisfy the same set of first-order sentences in the language of $S$). He then characterizes the elementarily $\kappa$-homogeneous structures $(A, <)$ where $A$ is a set, $<$ is a binary relation on $A$, and $\aleph_0 \leq \kappa \leq |A|$. In [15], the first author characterizes the elementarily $\kappa$-homogeneous structures $(A, f)$, where $f : A \to A$. Within group theory, Robinson and Timm call a group $G$ an *hc group* provided any two subgroups of $G$ of finite index are isomorphic ([20]). An abelian group with this property is called *minimal*, and minimal abelian groups have also received attention in the literature (see [11, 21]). In topology, a space $X$ is called a *Toronto space* if $X$ is homeomorphic to all its subspaces of the same cardinality. The countably infinite Toronto spaces have all been classified, but the question of whether there exists an uncountable nondiscrete Hausdorff Toronto space remains open (see [23] for details).

In this article, we consider a sort of dual to the concept of congruence of modules over a commutative ring. A fairly stringent notion already appears in the literature. In particular, Hirano and Mogami define a module $M$ over a (not necessarily commutative) ring $R$ to be *anti-Hopfian* provided $M$ is not simple and $M/N \cong M$ for every proper submodule $N$ of $M$ ([8]). Szélpál had already characterized such modules over $\mathbb{Z}$ in [24]. Hirano and Mogami extend Szélpál's result to modules over an arbitrary commutative ring (among other classes of rings). We recall their principal results over a commutative ring.

**Fact 1** ([8, Theorem 8 and Theorem 10]). *Let $R$ be a commutative ring, and let $M$ be a nonsimple $R$-module. Then $M$ is anti-Hopfian if and only if the lattice of submodules of $M$ is isomorphic to $\omega + 1$ (where $\omega$ is the first infinite ordinal) if and only if $M$ is Artinian and uniserial. In this case, the ring $S := End_R(M)$ is a complete discrete valuation ring and $M \cong K/S$, where $K$ is the quotient field of $S$.*

**Remark 1.** Uniserial modules are sometimes assumed to be Artinian in the literature (though this is not assumed in [8]). Uniserial modules which are not necessarily Artinian are often called generalized uniserial modules. For clarity, we remark that throughout this article, "uniserial module" denotes a module $M$ whose submodules are linearly ordered by set inclusion, but we do not assume that $M$ is Artinian.

We define an infinite module $M$ over a ring $R$ to be *homomorphically congruent* (HC for short) if and only if $M/N \cong M$ for every submodule $N$ of $M$ for which $|M/N| = |M|$. This definition also generalizes another notion studied by the authors in [18]. Specifically, an infinite module $M$ over a ring $R$ is said to be *homomorphically* *smaller* (HS for short) over $R$ if and only if $|M/N| < |M|$ for every nonzero submodule $N$ of $M$ (note that an HS $R$-module is trivially HC). We will make use of several of our results from [18] throughout the article.

The outline of this article is as follows. After providing several examples, we prove some general results on HC modules. Among other results, we prove that an HC module is either torsion or torsion-free, and then we classify the torsion-free HC modules. Further, HC module-theoretic characterizations of discrete valuation rings, almost Dedekind domains, and fields are given. We use these results to characterize the HC modules over a Dedekind domain, extending Scott's classification over $\mathbb{Z}$ in [22]. Finally, we close with some open questions.

## 2. EXAMPLES

We begin with a natural example which characterizes the HC vector spaces over a field.

**Example 1.** Let $F$ be a field, and let $V$ be an infinite $F$-vector space. Then the following are equivalent:

(a) $V$ is HC;
(b) $Dim(V) = 1$ or $|V| > |F|$.

*Proof.* Note first that if $0 \neq V$ is a vector space over a field $F$, then $|V| \geq |V/W| \geq |F|$ for every proper subspace $W$ of $V$.

(a) $\Rightarrow$ (b): Since $0 \neq V$, either $|V| > |F|$ or $|V| = |F|$. If $|V| = |F|$, then $|V| = |V/kerf|$, where $f$ is a projection of $V$ onto a one-dimensional subspace. As $V$ is HC, this implies that $V \cong V/kerf$. Hence $\dim(V) = \dim(V/kerf) = 1$.

(b) $\Rightarrow$ (a): If $\dim(V) = 1$, then $V$ is obviously HC. Suppose that $|V| > |F|$, and consider a subspace $W$ such that $|V/W| = |V| > |F|$. If $\dim(V)$ is finite, then $|V| = |F|^{dim(V)}$ and $|V/W| = |F|^{dim(V/W)}$. This implies that $\dim(V) = \dim(V/W)$ and hence $V \cong V/W$. Suppose now that $\dim(V)$ is infinite. Then since $|V| > |F|$, we see that $|V| = \max(|F|, \dim(V)) = \dim(V)$. If $\dim(V/W)$ is also infinite, then $\dim(V) = |V| = |V/W| = \max(|F|, \dim(V/W)) = \dim(V/W)$, and hence $V \cong V/W$. If $\dim(V/W)$ is finite, then $\dim(V) = |V| = |V/W| = |F|^{dim(V/W)}$. Since $\dim(V)$ is infinite, this implies that $|F|$ is infinite. But then we have $|V| = |F|^{dim(V/W)} = |F|$, a contradiction. □

The following corollary is easily established.

**Corollary 1.** *Every ring admits an HC module.*

*Proof.* Let $R$ be a ring, and let $J$ be a maximal ideal of $R$. Taking a sufficiently large direct sum of copies of $R/J$ yields an HC $R$-module by Example 1. □

In view of Example 1, all rings throughout the remainder of the article will be assumed not to be fields unless indicated otherwise.

Note that if $R$ is not a field, then the module constructed in Corollary 1 will be torsion but not faithful. We now present an example of a faithful torsion HC module.

**Example 2.** Suppose that $V$ is a discrete valuation ring with quotient field $K$. Then $K/V$ is an HC module over $V$.

*Proof.* We assume that $V$ is a DVR with quotient field $K$. Let $J = (v)$ be the maximal ideal of $V$. Every nonzero element of $V$ is of the form $uv^k$ for some unit $u$ and non-negative integer $k$. It follows that every nonzero element of $K$ is of the form $uv^m$ for some unit $u$ of $V$ and integer $m$. Suppose that $V \subseteq M \subsetneq K$ and that $M$ is a $V$-submodule of $K$. We claim that $(K/V)/(M/V) \cong K/V$. This is, of course, equivalent to $K/M \cong K/V$. Since $M \neq K$, there exists a least integer $k$ such that $v^k \in M$. Define the function $\varphi : K \to K/V$ by $\varphi(x) := v^{-k}x \pmod V$. It is easy to see that $\varphi$ is $V$-linear and onto $K/V$. Now let $x \in K$. Note that $x \in Ker(\varphi)$ if and only if $v^{-k}x \in V$ if and only if $x \in Vv^k$ if and only if $x \in M$. Thus $K/M \cong K/V$ and the proof is complete. □

**Remark 2.** Note that the above proof shows that not only is $K/V$ HC, but that $K/V$ has the stronger anti-Hopfian property. This fact also appears explicitly in [8].

Recall that Scott characterized the HC modules over the integers in [22]. Before stating his result, we remind the reader that the *quasi-cyclic group of type p* ($p$ prime) is the direct limit of the groups $\mathbb{Z}/(p^n)$. We denote this group by $C(p^\infty)$.

**Example 3** (Scott, [22]). Let $G$ be an infinite abelian group. Then $G$ is HC if and only if $G$ belongs to one of the following families:

(a) $\mathbb{Z}$;
(b) $\bigoplus_\kappa \mathbb{Z}/(p)$, where $\kappa \geq \aleph_0$;
(c) $\bigoplus_\kappa C(p^\infty)$, where $\kappa = 1$ or $\kappa > \aleph_0$.

We now present examples of domains which are HC as modules over themselves. Recall from the introduction that a module $M$ is HS if and only if $|M/N| < |M|$ for every nonzero submodule $N$ of $M$.

**Example 4.** Let $\mathbb{F}$ be a finite field. Then the rings $\mathbb{F}[x]$ and $\mathbb{F}[[t]]$ are HS as modules over themselves.

*Proof.* Assume $\mathbb{F}$ is a finite field. It is well known (and easy to show via the division algorithm) that $\mathbb{F}[x]/(f(x))$ is finite if $f(x) \neq 0$. Thus if $|\mathbb{F}[x]/(f(x))| = |\mathbb{F}[x]|$, then $f(x) = 0$. Hence $\mathbb{F}[x]$ is HS. Now consider the power series ring $R := \mathbb{F}[[t]]$. It is also well known that $R$ is a DVR and that $\{(t^n) : n > 0\}$ is the set of proper nonzero ideals of $R$. It is clear that $\mathbb{F}[[t]]/(t^n)$ is finite for every positive integer $n$, and hence $\mathbb{F}[[t]]$ is also HS. □

Non-Noetherian rings also exist which are HS as modules over themselves.

**Example 5.** For every uncountable cardinal $\kappa$, there exists a non-Noetherian valuation domain $V$ such that $|V/I| < |V|$ for every nonzero ideal $I$ of $V$.

*Proof.* The existence of such a ring $V$ is established in Theorem 2.8 of [18]. □

## 3. MAIN RESULTS

In this section, we develop the general theory of HC modules. We begin with the following useful proposition.

**Proposition 1.** *Suppose that $M$ is an HC module over the ring $R$. Then the annihilator of $M$, $Ann(M)$, is a prime ideal of $R$.*

*Proof.* We assume that $M$ is an HC module over $R$. Let $Ann(M)$ be the annihilator of $M$. We suppose that $r, s \in R$ and that $r \notin Ann(M)$ and $s \notin Ann(M)$. We will prove that $rs \notin Ann(M)$. We first claim the following:

$$|M/rM| < |M|, \tag{3.1}$$

$$|M/sM| < |M|. \tag{3.2}$$

We establish only (3.1) as (3.2) follows analogously. Suppose by way of contradiction that $|M/rM| = |M|$. Since $M$ is HC, it follows that $M/rM \cong M$. But note that $r$ annihilates $M/rM$, and hence $r$ annihilates $M$. This contradicts $r \notin Ann(M)$. Thus (3.1) and (3.2) are established.

Let $\{m_i : i \in \alpha\}$ be a complete set of coset representatives for $M$ mod $sM$. Then $\{rm_i : i \in \alpha\}$ contains a set of coset representatives for $rM$ mod $rsM$. Hence,

$$|rM/rsM| \leq \alpha = |M/sM| < |M|. \tag{3.3}$$

We now suppose by way of contradiction that $rs \in Ann(M)$. Then $rsM = \{0\}$. But then (3.3) above implies that $|rM| < |M|$. However, since $M$ is infinite and $|rM| < |M|$, it follows from elementary cardinal arithmetic that $|M/rM| = |M|$. This contradicts (3.1), and the proof is complete. □

We recall that an element $m$ in an $R$-module $M$ is *torsion*, provided $rm = 0$ for some nonzero $r \in R$. If every element of $M$ is torsion, then $M$ is called a *torsion module*. If only $0 \in M$ is torsion, then $M$ is *torsion-free*. All examples of HC modules presented thus far have either been torsion or torsion-free. We prove that this is always the case.

**Theorem 1.** *Every HC module is either torsion or torsion-free.*

*Proof.* Suppose that $M$ is HC over the ring $R$, and let $Ann(M)$ be the annihilator of $M$ in $R$. If $Ann(M)$ is nonzero, then of course $M$ is torsion, and we are done. Thus we assume that $Ann(M) = \{0\}$. Proposition 1 implies that $\{0\}$ is a prime ideal of $R$, and hence $R$ is a domain. Since $R$ is a domain, the set $T$ of torsion elements of $M$ forms a submodule of $M$.

We suppose first that $|M/T| = |M|$. Since $M$ is HC, it follows that $M/T \cong M$, and thus $M$ is torsion-free.

We now assume that $|M/T| < |M|$. Let $N$ be a submodule of $M$ maximal with respect to the property of having no nonzero torsion elements (such an $N$ exists by Zorn's Lemma). We claim that $M/N$ is torsion. Hence suppose that $m \in M - N$. By maximality of $N$, $(N, m)$ contains a nonzero torsion element, say $n + rm$ (note that $r \neq 0$). Thus there exists some nonzero $x \in R$ such that $xn = (-xr)m$. Hence $(-xr)m \in N$, and it follows that $M/N$ is torsion. Recall our assumption above that $|M/T| < |M|$. Since $M$ is infinite, we deduce that $|T| = |M|$. Note that the map $t \mapsto N + t$ is an injective mapping from $T$ into $M/N$. It follows that $|M| = |T| \leq |M/N| \leq |M|$. Thus $|M/N| = |M|$. Since $M$ is HC and since $M/N$ is torsion, $M$ must also be torsion. This completes the proof. □

We turn our attention toward describing the torsion-free HC modules. We first state some results on HS modules from [18] which will be of use to us.

**Fact 2.** *Let $D$ be a domain, let $K$ be the quotient field of $D$, and let $M$ be an infinite, faithful module over $D$. Then the following hold*:

(a) *If $M$ is HS over $D$ and $N$ is a nonzero submodule of $M$, then $N$ is also HS over $D$.*
(b) *$M$ is HS over $D$ if and only if $D$ is an HS domain (that is, $D$ is HS as a module over itself), $D \subseteq M \subseteq K$ (up to isomorphism), and $|M/D| < |D|$.*
(c) *In case $D$ is countable, then $M$ is HS over $D$ if and only if $D$ is a one-dimensional Noetherian domain with all residue fields finite and $M$ is isomorphic to a nonzero ideal of $D$.*

*Proof.* These results are stated and proved in Lemma 3.1, Theorem 3.3, and Theorem 4.2 of [18], respectively. □

We now show that the torsion-free HC modules are precisely the HS modules. Using Fact 2, this gives us a description of the class of torsion-free HC modules.

**Theorem 2.** *Let $R$ be a ring and suppose that $M$ is a nontrivial torsion-free module over $R$ (hence $R$ is a domain). Then $M$ is HC if and only if $M$ is HS.*

*Proof.* We assume that $R$ is a ring and that $M$ is a nontrivial torsion-free HC $R$-module. Then $R$ is a domain, $M$ is infinite, and $M$ is faithful over $R$. We prove that $M$ is homomorphically smaller (the converse is immediate). Since $R$ is not a field (recall our assumption following Example 1), there exists some nonzero element $x \in R$ which is not invertible. Now let $N$ be an arbitrary nonzero submodule of $M$ and let $n$ be a nonzero element of $N$. We claim that $|M/(xn)| < |M|$. For suppose by way of contradiction that $|M/(xn)| = |M|$. Since $M$ is HC, we conclude that $M \cong M/(xn)$. Since $x$ is not invertible and $M$ is torsion-free, it follows that $n \notin (xn)$. But then $\overline{n}$ is a nonzero torsion element of $M/(xn)$, yet $M$ is torsion-free. This contradicts $M \cong M/(xn)$, and hence $|M/(xn)| < |M|$. Now simply observe that as $(xn) \subseteq N$, we infer that $|M/N| \leq |M/(xn)| < |M|$, whence $|M/N| < |M|$. We have shown that $M$ is homomorphically smaller, and the proof is complete. □

With the previous results in hand, we turn our attention toward describing the uniserial HC modules. This is a natural question given that the definition of an HC module generalizes the definition of an anti-Hopfian module and (by Fact 1) all anti-Hopfian modules are uniserial. Note that by modding out the annihilator, there is no loss of generality in restricting to faithful modules over a domain by Proposition 1.

**Theorem 3.** *Let $M$ be an infinite faithful uniserial module over the domain $D$, and let $K$ be the quotient field of $D$. Then $M$ is HC if and only if one of the following holds*:

(i) *$D$ is an HS (valuation) domain, $D \subseteq M \subseteq K$, and $|M/D| < |D|$;*
(ii) *There exists an anti-Hopfian submodule $N$ of $M$ such that $M/L \cong M$ for all submodules $L \subsetneq N$, and $|M/L| < |M|$ for all submodules $L$ containing $N$.*

*Proof.* Assume that $M$ is an infinite faithful uniserial module over the domain $D$, and let $K$ be the quotient field of $D$. If (i) holds, then $M$ is HC by Fact 2. If (ii) holds, then (using the fact that $M$ is uniserial) it is clear that $M$ is HC.

Conversely, assume that $M$ is HC. We suppose first that $M$ is HS. Then (b) of Fact 2 implies that (i) holds.

We now suppose that $M$ is not HS, and we show that (ii) holds. Since $M$ is not HS, there exists a nonzero submodule $P$ of $M$ such that

$$|M/P| = |M|. \tag{3.4}$$

Now fix a nonzero element $x \in P$. By Zorn's Lemma, there exists a submodule $S$ of $P$ which is maximal with respect to not containing $x$. Thus any submodule of $P$ which properly contains $S$ also contains $x$. In fact, a much stronger statement holds:

Every submodule of $M$ which properly contains $S$ also contains $x$. (3.5)

Indeed, suppose $T$ is a submodule of $M$ which properly contains $S$. Since $M$ is uniserial, either $P \subseteq T$ or $T \subseteq P$. If $P \subseteq T$, then since $x \in P$, also $x \in T$. Thus assume that $T \subseteq P$. Since $T$ properly contains $S$ and $T \subseteq P$, it follows by maximality of $S$ that $x \in T$, and (3.5) is established.

Recall from (3.4) that $|M/P| = |M|$. As $S \subseteq P$, we have $|M| = |M/P| \leq |M/S| \leq |M|$. Hence $|M/S| = |M|$ as well. Since $M$ is HC, we deduce that $M/S \cong M$. Statement (3.5) implies that $M/S$ contains a cyclic minimum submodule, and hence the same is true of $M$. Let $Dx_1$ be the minimum submodule of $M$. Since $M$ is uniserial and $Dx_1$ is minimal, it follows that $Dx_1 \subseteq P$. This fact along with Eq. 3.4 above implies that $|M/Dx_1| = |M|$, whence $M \cong M/Dx_1$. Since $M$ has a minimum cyclic submodule, the same is true of $M/Dx_1$. By induction, we obtain a sequence of elements $x_1, x_2, x_3, \ldots$ of $M$ such that

$$\{x_0 := 0\} \subsetneq Dx_1 \subsetneq Dx_2 \subsetneq Dx_3 \cdots \tag{3.6}$$

and for each $i$, $Dx_{i+1}/Dx_i$ is simple. Let $N := \bigcup_{i\in\mathbb{N}} Dx_i$. We claim that $N$ is Artinian. To prove this, it clearly suffices to show that

The $Dx_i$ are precisely the proper submodules of $N$. (3.7)

Let $n \in N$ be arbitrary, and let $i$ be least such that $n \in Dx_i$. We claim that $Dn = Dx_i$. This is obvious if $i = 0$, so assume $i > 0$. Since $n \notin Dx_{i-1}$, we conclude (from the fact that $M$ is uniserial) that $Dx_{i-1} \subsetneq Dn$. Thus $Dx_{i-1} \subsetneq Dn \subseteq Dx_i$. Since $Dx_i/Dx_{i-1}$ is simple, we are forced to conclude that $Dn = Dx_i$, as claimed. It is now clear that the $Dx_i$ are the only proper submodules of $N$, and hence $N$ is Artinian. Fact 1 implies that $N$ is anti-Hopfian. It follows from the induction and (3.7) above that $M \cong M/L$ for all submodules $L \subsetneq N$.

To finish the proof of (ii), it clearly suffices to show that $|M/N| < |M|$. Suppose by way of contradiction that $|M/N| = |M|$. Since $M$ is HC and $M$ has a mimimum submodule, the same is true of $M/N$. Thus there is a submodule $U$ of $M$ properly containing $N$ such that $U/N$ is simple. Since $N$ is Artinian and $U$ is uniserial, it is easy to see that $U$ is also Artinian. We may now invoke Fact 1 to conclude that $U$ is anti-Hopfian. However, the lattice of submodules of $U$ is isomorphic to $\omega + 2$, contradicting Fact 1. This contradiction shows that $|M/N| < |M|$, and completes the proof. □

**Remark 3.** Consider the anti-Hopfian submodule $N$ in part (ii) of Theorem 3. Note that, in a sense, $M$ is anti-Hopfian below $N$ and HS above $N$, and thus $N$ serves as a sort of boundary between these two conditions. We do not know if it is possible for $N$ to be a *proper* submodule of $M$. We will demonstrate that if the operator domain is Noetherian, then this is not possible.

We will shortly obtain a characterization of the uniserial HC modules over a Noetherian ring. We first present two useful lemmas.

**Lemma 1** ([13, Lemma 3]). *Let $R$ be a ring and let $I$ be a finitely generated ideal of $R$. Then*:

(1) *If $R/I$ is finite, then $R/I^n$ is finite for all positive integers $n$*;
(2) *If $R/I$ has infinite cardinality $\kappa$, then so does $R/I^n$ for all positive integers $n$.*

**Lemma 2.** *Suppose that $M$ is a uniserial module over the Noetherian ring $R$. Then every proper submodule of $M$ is cyclic.*

*Proof.* Let $M$ and $R$ be as stated, and suppose that $N$ is a proper submodule of $M$. Choose any $m \in M - N$. Then as $M$ is uniserial, it follows that $N \subseteq Rm$. Now, $Rm \cong R/I$, where $I$ is the annihilator of $m$. Since $R$ is Noetherian, $R/I$ is also a Noetherian $R$-module. As $N$ can be embedded into $R/I$, we deduce that $N$ is finitely generated. Since $M$ is uniserial, we conclude that $N$ is cyclic. □

We now characterize the uniserial HC modules over a Noetherian ring.

**Proposition 2.** *Let $D$ be a Noetherian domain with quotient field $K$, and let $M$ be an infinite faithful module over $D$. Then $M$ is a uniserial HC $D$-module if and only if one of the following holds*:

(1) *$(D, J)$ is a countable discrete valuation ring (DVR; $J$ denotes the maximal ideal of $D$), $D/J$ is finite, and $M$ is isomorphic to $J^n$ for some positive integer $n$*;

(2) $(D, J)$ *is an uncountable DVR,* $|D/J| < |D|$, *and* $M$ *is isomorphic to a D-submodule of* $K$;
(3) $M$ *is anti-Hopfian.*

*Proof.* Let $D$, $K$, and $M$ be as stated. Note first that the modules satisfying (1) or (2) are trivially uniserial. As for (3), all anti-Hopfian modules are uniserial by Fact 1. We now show that the modules in (1)–(3) are HC.

Suppose first that $(D, J)$ is a countable DVR, $D/J$ is finite, and $M$ is isomorphic to $J^n$ for some positive integer $n$. Lemma 1 implies that $D/J^i$ is finite for every positive integer $i$, and thus $D$ is an HS domain (recall that the ideals $J^i$ are the only proper nonzero ideals of $D$). Part (a) of Fact 2 implies that $M$ is also HS, and hence $M$ is HC.

Suppose now that $(D, J)$ is an uncountable DVR, $|D/J| < |D|$, and $M$ is isomorphic to a $D$-submodule of $K$. Lemma 1 implies that $|D/J^i| < |D|$ for every positive integer $i$, and hence again, $D$ is an HS domain. We will show that $M$ is also HS. To do this, it suffices by (a) of Fact 2 to prove that $K$ is an HS module over $D$. Toward this end, it suffices by (b) of Fact 2 to prove that $|K/D| < |D|$. To see this, note that $K/D \cong \lim_{i\to\infty} D/J^i$. Lemma 1 implies that $|K/D| = |\lim_{i\to\infty} D/J^i| = \max(\aleph_0, |D/J|) < |D|$, and thus (2) holds.

As for (3), if $M$ is anti-Hopfian, $M$ is trivially HC.

Conversely, suppose that $M$ is a uniserial HC module over the Noetherian domain $D$. We will show that (1), (2), or (3) holds. Suppose first that $M$ is HS. Then Fact 2 shows that (since $D \subseteq M$ up to isomorphism) $D$ is a DVR and (1) or (2) holds. Otherwise, Theorem 3 implies that $M$ possesses an anti-Hopfian submodule $N$. Since $N$ is not finitely generated, we infer from Lemma 2 that $N$ cannot be proper. Hence $N = M$, and $M$ is anti-Hopfian. □

**Remark 4.** DVRs satisfying (1) and (2) of Proposition 2 are not hard to come by. Indeed, let $\kappa$ be a cardinal of countable cofinality. König's Theorem implies that $\kappa^{\aleph_0} > \kappa$. Now let $F$ be a field of cardinality $\kappa$. Then the power series ring $F[[t]]$ is a DVR of cardinality $\kappa^{\aleph_0}$ with residue field (isomorphic to) $F$ of cardinality $\kappa$.

Next we consider Noetherian and Artinian HC modules.

**Theorem 4.** *Let* $M$ *be an infinite faithful Noetherian module over the domain* $D$. *Then* $M$ *is HC if and only if* $D$ *is a Noetherian HS domain and* $M$ *is isomorphic to a nonzero ideal of* $D$.

*Proof.* Let $M$ be an infinite faithful Noetherian module over the domain $D$. We suppose that $D$ is a Noetherian HS domain and that $M$ is isomorphic to a nonzero ideal of $D$. Then $M$ is also HS over $D$ by Fact 2, whence $M$ is HC. Conversely, suppose that $M$ is HC. We first show that $M$ is HS. Suppose not. Then there is a nonzero submodule $N_1$ of $M$ such that $|M| = |M/N_1|$. Since $M$ is HC, $M \cong M/N_1$. But then there is a submodule $N_2$ properly containing $N_1$ such that $M/N_1 \cong (M/N_1)/(N_2/N_1) \cong M/N_2$. Continuing inductively, we obtain a strictly increasing chain

$$N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \cdots$$

of submodules of $M$. This contradicts the fact that $M$ is Noetherian, and we conclude that $M$ is HS. By (b) of Fact 2, $D$ is an HS domain and (up to isomorphism) $D \subseteq M \subseteq K$, where $K$ is the quotient field of $D$. Since $M$ is finitely generated, it is easy to see that $M$ is isomorphic to some nonzero ideal $I$ of $D$.

It remains to show that $D$ is Noetherian. Toward this end, let $J \subseteq D$ be an arbitrary ideal. We will show that $J$ is finitely generated. Since $M$ is Noetherian and $M \cong I$, it follows that every ideal $J$ contained in $I$ is finitely generated. Let $x \in I$ be nonzero. Then $Jx \subseteq I$, and hence $Jx$ is a finitely generated ideal. As $J \cong Jx$, we deduce that $J$ is finitely generated. Thus $D$ is Noetherian, and the proof is complete. □

**Theorem 5.** *Let $M$ be an infinite faithful Artinian module over the domain $D$. Suppose further that the socle of $M$ is simple. Then $M$ is HC if and only if $M$ is anti-Hopfian.*

*Proof.* We suppose that $M$ is an infinite faithful HC Artinian module with simple socle over the domain $D$. We will show that $M$ is anti-Hopfian (the converse is patent). We recall that if $J$ is a maximal ideal of $D$, then the $J$-component $M[J]$ of $M$ is defined by $M[J] := \{m \in M : J^n m = 0 \text{ for some positive integer } n\}$. Since $M$ is Artinian, it is well known (see Lemma 1.7 of [25] for instance) that $M = \bigoplus_{i=1}^{n} M[J_i]$ for some maximal ideals $J_1, J_2, \ldots, J_n$ of $D$. By elementary cardinal arithmetic, it follows that $|M[J_i]| = |M|$ for some $i$. Since $M$ is HC, we infer that $M \cong M[J_i]$ for some $i$. Hence

$$M \text{ is } J\text{-primary for some maximal ideal } J \text{ of } D. \tag{3.8}$$

We now claim that

$$M \text{ is not HS.} \tag{3.9}$$

If $M$ were HS, then Fact 2 would imply that $M$ is torsion-free. Since $M$ is also Artinian, this implies that $D$ is a field, contradicting our assumption (recall the comments following Example 1). Hence (3.9) is established.

Since the socle $S$ of $M$ is simple, $S = Dx_1$ for some nonzero $x_1 \in M$. Let $K$ be an arbitrary nonzero submodule of $M$. Since $M$ is Artinian, $K$ contains a minimal submodule. The simplicity of $S$ implies that $S = Dx_1 \subseteq K$. We have shown that

$$Dx_1 \subseteq K \text{ for all nonzero submodules } K \text{ of } M. \tag{3.10}$$

Recall from (3.9) above that $M$ is not HS. Thus $|M/L| = |M|$ for some nonzero submodule $L$ of $M$. It follows from (3.10) that $Dx_1 \subseteq L$, and hence $|M| = |M/L| \leq |M/Dx_1| \leq |M|$. Thus $|M/Dx_1| = |M|$. Since $M$ is HC, we obtain (by induction) a sequence of elements $x_1, x_2, x_3, \ldots$ of $M$ such that

$$\{x_0 := 0\} \subsetneq Dx_1 \subsetneq Dx_2 \subsetneq Dx_3 \cdots \tag{3.11}$$

and for each $i$, if $X$ is any submodule of $M$ properly containing $Dx_i$, then $X$ contains $Dx_{i+1}$. Let $N := \bigcup_{i \in \mathbb{N}} Dx_i$. Then it follows (as in the proof of Theorem 3) that

$$\text{the submodules } Dx_i \text{ are exactly the proper submodules of } N, \tag{3.12}$$

whence $N$ is uniserial.

To complete the proof that $M$ is anti-Hopfian, it suffices (by Fact 1) to show that $N = M$. It follows from (3.8), (3.11), and (3.12) that

$$Jx_{i+1} = Dx_i \tag{3.13}$$

for all $i \in \mathbb{N}$. Suppose by way of contradiction that $N \neq M$, and let $m \in M - N$. An easy induction shows that $Dx_i \subseteq Dm$ for every $i \in \mathbb{N}$, and thus $N \subseteq Dm$. Since $M$ is $J$-primary, it follows that $J^n m = 0$ for some positive integer $n$, and thus also $J^n N = \{0\}$. However, (3.13) shows that this is impossible. We conclude that $N = M$, and hence $M$ is anti-Hopfian. □

**Remark 5.** We do not know if we need the assumption that the socle of $M$ is simple to deduce that $M$ is anti-Hopfian.

We finish this section by using our results to give HC module-theoretic characterizations of discrete valuation rings, almost Dedekind domains, and fields.

**Proposition 3.** *Let $D$ be a domain with quotient field $K$. Then $D$ is a DVR if and only if $K/D$ is anti-Hopfian.*

*Proof.* Assume that $D$ is a domain with quotient field $K$. Suppose first that $D$ is a DVR. We showed in Example 2 that $K/D$ is anti-Hopfian. Conversely, suppose that $K/D$ is anti-Hopfian. Fact 1 states that the lattice of submodules of $K/D$ (under inclusion) is isomorphic to $\omega + 1$, where $\omega$ is the first infinite ordinal. Hence $K/D$ is uniserial. This implies that $D$ is a valuation domain (this is well known, but we include the short proof). To see this, let $x$ and $y$ be arbitrary nonzero elements of $D$. Since $K/D$ is uniserial, it follows that either $\frac{D}{x} \subseteq \frac{D}{y}$, or $\frac{D}{y} \subseteq \frac{D}{x}$. Without loss of generality, we may assume that $\frac{D}{x} \subseteq \frac{D}{y}$. In particular, $\frac{1}{x} \in \frac{D}{y}$. It follows that $y \in Dx$, and hence $D := V$ is a valuation domain. The fact that the lattice of submodules of $K/V$ is order-isomorphic to $\omega + 1$ implies that the value group of $V$ is isomorphic to $\mathbb{Z}$, whence $V$ is a DVR. □

We can obtain a local version of the previous result to characterize the almost Dedekind domains. Recall that a domain $D$ is an *almost Dedekind domain* if and only if $D$ is locally a discrete valuation ring; that is, if and only if $D_J$ is a DVR for every maximal ideal $J$ of $D$. We omit the straightforward proof of the following corollary.

**Corollary 2.** *Let $D$ be a domain with quotient field $K$. Then $D$ is an almost Dedekind domain if and only if $K/D_J$ is an anti-Hopfian $D$-module for every maximal ideal $J$ of $D$.*

Before stating the final proposition of this section, we recall that an $R$-module $M$ is *large* (more precisely, *$R$-large*) provided $M$ is infinite and $|M| > |R|$.

**Proposition 4.** *Let $R$ be a ring. The following are equivalent*:

(a) *$R$ is a field*;
(b) *Every $R$-module may be embedded into an HC $R$-module*;
(c) *$R$ admits a large torsion-free HC $R$-module*;

(d) *Every large R-module is HC*;
(e) *Every R-module is the homomorphic image of an HC R-module.*

*Proof.* It follows easily from Example 1 that (a) $\Rightarrow$ (b)–(e). We now show that each of (b)–(e) implies (a).

(b) $\Rightarrow$ (a): Suppose by way of contradiction that every $R$-module may be embedded into an HC $R$-module but that $R$ is not a field. Then $R$ possesses a proper nonzero ideal $I$. By assumption, $R \oplus R/I$ may be embedded into an HC $R$-module $M$. Theorem 1 implies that $M$ is either torsion or torsion-free. Thus also $R \oplus R/I$ is torsion or torsion-free, which is false.

(c) $\Rightarrow$ (a): Assume by way of contradiction that $R$ admits a large torsion-free HC module $M$ but that $R$ is not a field. Since $R$ admits a torsion-free module, it follows that $R$ is a domain. Fact 2 and Theorem 2 imply that $|M| = |R|$, contradicting that $M$ is large.

(d) $\Rightarrow$ (a): Suppose by way of contradiction that every large $R$-module is HC and that $R$ is not a field. Let $\kappa$ be an infinite cardinal larger than $|R|$. Then by assumption, $\bigoplus_\kappa R$ is HC. Since $R$ has an identity, $\bigoplus_\kappa R$ is not torsion, and hence it must be torsion-free by Theorem 1. It follows that $R$ is a domain. Now let $x \in R$ be a nonzero nonunit. Then the same is true of $x^2$. Consider $M := \bigoplus_\kappa R/(x^2)$. By our assumption, $M$ is HC. Note that $(x^2) = Ann(M)$. Since $Ann(M)$ is prime (Proposition 1), it follows that $x \in Ann(M)$. But then $x \in (x^2)$. Since $R$ is a domain, this implies that $x$ is a unit, a contradiction.

(e) $\Rightarrow$ (a): Assume every module is the homomorphic image of an HC $R$-module, and let $\kappa > |R|$. Then in particular, $\bigoplus_\kappa R$ is the homomorphic image of an HC $R$-module $M$. By Theorem 1, $M$ is either torsion or torsion-free. If $M$ is torsion, then so is $\bigoplus_\kappa R$, but this is false since $R$ has an identity. Thus $M$ is torsion-free, and so $R$ is a domain. Let $K$ be the quotient field of $R$. If $R$ is not a field, then Fact 2 and Theorem 2 imply that $R \subseteq M \subseteq K$. But then $|M| = |R|$, and so it is not possible for $\bigoplus_\kappa R$ to be a homomorphic image of $M$. We conclude that $R$ is a field, and the proof is complete. $\square$

## 4. HC MODULES OVER A DEDEKIND DOMAIN

We begin this section with a proposition which yields further examples of HC modules.

**Proposition 5.** *Suppose that $D$ is a countable HS domain, and assume that $R$ is a finite integral extension of $D$ (that is, $R$ is integral over $D$ and $R$ is finitely generated as a $D$-module). Then $R$ is also an HS domain.*

*Proof.* We assume that $D$ and $R$ are as stated. By (c) of Fact 2, we need only show that $R$ is one-dimensional Noetherian domain with all residue fields finite. Since $R$ is integral over $D$ and $D$ is one-dimensional Noetherian, $R$ inherits these properties by integrality (we refer the reader to Chapter 2 of [3] for a comprehensive treatment of integral dependence). We now show that all residue fields of $R$ are finite. Let $J$ be a maximal ideal of $R$, and let $J^c$ be the contraction of $J$ to $D$. Then $J^c$ is maximal in $D$ (again, by integrality) and (as $R$ is a finite integral extension of $D$) $R/J$ is a finite field extension of $D/J^c$. Since $D/J^c$ is finite, so is $R/J$. This completes the proof. □

The next example follows immediately from the previous proposition.

**Example 6.** The ring $\mathbb{Z}[\sqrt{10}]$ is a Dedekind HS domain which is not a PID.

We now proceed to classify the HC modules over an arbitrary Dedekind domain, extending Scott's results over $\mathbb{Z}$ in [22]. The classical definition of a Dedekind domain is a domain for which every (proper, nonzero) ideal factors as a product of prime ideals. We will make use of the following definition.

**Definition 1.** Let $D$ be a domain with quotient field $K$, $P$ a nonzero prime ideal of $D$, and $M$ a $D$-module. The $P$-component of $M$ is defined to be the submodule of $M$ consisting of the elements of $M$ which are killed by a power of $P$. In case $M = K/D$ and $D$ is a Dedekind domain, the $P$-component of $M$ is denoted by $C(P^\infty)$.

Recall that every HC module is either torsion or torsion-free (Theorem 1). The following proposition classifies the torsion HC modules over an arbitrary Dedekind domain (which is not a field). The proof follows more or less *mutatis mutandis* by mimicking the proof of the classification of such modules over $\mathbb{Z}$ given by Scott in [22] (by using Kaplansky's [9], which carries over many fundamental abelian group-theoretic results to modules over a Dedekind domain). As such, we omit it.

**Proposition 6.** *Let $D$ be a Dedekind domain, and let $M$ be an infinite $D$-module. Then $M$ is a torsion HC module over $D$ if and only if one of the following holds*:

(1) $M \cong \bigoplus_\kappa D/P$ *for some maximal ideal $P$ of $D$ and cardinal $\kappa$. Further, either $\kappa = 1$ (and $D/P$ is infinite), or $\kappa$ is infinite and $\kappa > |D/P|$*;
(2) $M \cong \bigoplus_\kappa C(P^\infty)$ *for some maximal ideal $P$ of $D$ and cardinal $\kappa$. Further, either $\kappa = 1$ or $\kappa > |C(P^\infty)|$.*

Recall from Example 3 that the only torsion-free HC module (up to isomorphism) over the ring $\mathbb{Z}$ of integers is $\mathbb{Z}$ itself. We will see that the classification of the torsion-free HC modules over an arbitrary Dedekind domain is, in general, much more complicated. It is this task which we now take up. We will use the following lemma.

**Lemma 3.** *Let $D$ be a Dedekind domain. Suppose that $|D/J| < |D|$ for every maximal ideal $J$ of $D$. Then $|D/I| < |D|$ for every nonzero ideal $I$ of $D$.*

*Proof.* Assume that $D$ is a Dedekind domain such that $|D/J| < |D|$ for every maximal ideal $J$ of $D$. We will show the same is true for every nonzero ideal $I$ of $D$. Thus let $I$ be an arbitrary nonzero ideal of $D$. The result is obvious if $I = D$, so assume that $I$ is proper. Since $D$ is Dedekind, $I = P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k}$ for some nonzero prime (hence maximal) ideals $P_1, \ldots, P_k$ of $D$ and positive integers $n_1, \ldots, n_k$. It follows that

$$D/I = D/(P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k}) \cong D/P_1^{n_1} \times D/P_2^{n_2} \times \cdots \times D/P_k^{n_k}. \tag{4.1}$$

By assumption, $|D/P_i| < |D|$ for each $i$. Lemma 1 implies that $|D/P_i^{n_i}| < |D|$ for each $i$. It follows that $|D/I| < |D|$. □

We now complete our description of the HC modules over a Dedekind domain.

**Proposition 7.** *Let $D$ be a Dedekind domain, and suppose that $M$ is a nontrivial torsion-free module over $D$. Let $K$ be the quotient field of $D$. Then*:

(1) *$M$ is HC over $D$ if and only if every residue field of $D$ has smaller cardinality than $D$, $D \subseteq M \subseteq K$, and $|M/D| < |D|$.*
(2) *Moreover, if $D$ is countable, then $M$ is HC over $D$ if and only if all residue fields of $D$ are finite and $M$ is isomorphic (as a $D$-module) to a nonzero ideal of $D$.*

*Proof.* Immediate from Fact 2, Theorem 2, and the previous lemma. □

Several remarks are now in order. Let $D$ be a Dedekind domain. Proposition 6 shows that $D$ admits the faithful torsion HC module $C(P^\infty)$, where $P$ is a maximal ideal of $D$. It is clear that $D$ need not admit a torsion-free HC module. Indeed, let $D = F[x]$, the polynomial ring in one variable over an infinite field $F$. It is easy to see that $F$ embeds into $D/I$ for every proper ideal $I$ of $D$, whence $|F| \leq |D/J| \leq |D| = |F|$ for every maximal ideal $J$ of $D$ (thus $|D/J| = |D|$). Part (1) of Proposition 7 implies that $D$ does not admit a torsion-free HC module. Now let $\rho$ be an infinite cardinal. One may ask if there necessarily exists a Dedekind domain $D$ of cardinality $\rho$ which admits a torsion-free HC module. The answer, in general, is no. To see this, we first state a result from an earlier paper.

**Fact 3** ([18, Proposition 2.4]). *Let $\rho$ be an infinite cardinal. There exists a Noetherian HS domain $D$ (which is not a field) of cardinality $\rho$ if and only if there exists a cardinal $\kappa$ such that $\kappa < \rho \leq \kappa^{\aleph_0}$.*

Recall that a cardinal $\rho$ is a *strong limit* provided that whenever $\lambda < \rho$, then also $2^\lambda < \rho$ (the $\omega$th beth cardinal $\beth_\omega$ is a strong limit, for example). Using the above fact along with Theorem 2, we obtain the following corollary.

**Corollary 3.** *Suppose that $\rho$ is a strong limit and $D$ is a Noetherian domain of cardinality $\rho$. Then $D$ does not admit a torsion-free HC module.*

Despite this corollary, there exist arbitrarily large cardinals $\rho$ for which the following holds: For every positive integer $n$, there exists a principal ideal HS domain $D$ of cardinality $\rho$ with exactly $n$ maximal ideals. This follows immediately from Theorem 2.6 of [10], for example.

Lastly, we remark that if $D$ is a countable Dedekind domain, then Proposition 7 shows that all torsion-free HC modules over $D$ lie 'below' $D$ (up to isomorphism) in the sense that they are isomorphic to ideals of $D$ (as is the case with $\mathbb{Z}$). This is not necessarily the case if $D$ is uncountable. For example, let $D := \mathbb{F}_2[[t]]$, the power series ring in one variable over the field of two elements, and let $K$ be the quotient field of $D$. Then $D$ is a DVR with (unique) residue field $\mathbb{F}_2$, and $K/D \cong \varinjlim D/(t^n)$, which is countable. Proposition 7 implies that $K$ is an HC module over $D$, yet $K$ is not isomorphic (as a $D$-module) to an ideal of $D$.

## 5. OPEN QUESTIONS

We would like to know the answers to the following questions.

**Question 1.** Let $D$ be a domain with quotient field $K$. Can one give necessary and sufficient conditions in order for $K/D$ to be HC? In particular, must $D$ be a discrete valuation ring?

**Question 2.** What are necessary and sufficient conditions in order for a domain to admit a faithful torsion HC module?

**Question 3.** Is a faithful Artinian HC module necessarily anti-Hopfian?

**Question 4.** Is an indecomposable torsion HC module necessarily anti-Hopfian?

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