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On Modules Whose Proper Homomorphic Images Are of Smaller Cardinality

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Abstract. Let R be a commutative ring with identity, and let M be a unitary module over R. We call M H-smaller (HS for short) if and only if M is infinite and $|M/N| < |M|$ for every nonzero submodule N of M. After a brief introduction, we show that there exist nontrivial examples of HS modules of arbitrarily large cardinality over Noetherian and non-Noetherian domains. We then prove the following result: suppose M is faithful over R, R is a domain (we will show that we can restrict to this case without loss of generality), and K is the quotient field of R. If M is HS over R, then R is HS as a module over itself, $R \subseteq M \subseteq K$, and there exists a generating set S for M over R with $|S| < |R|$. We use this result to generalize a problem posed by Kaplansky and conclude the paper by answering an open question on Jónsson modules.

1 Introduction

Throughout, all rings are assumed commutative with identity and all modules are assumed unitary.

Kaplansky posed the problem of showing that $\mathbb{Z}$ is the unique infinite abelian group G with the property that G/H is finite for every nonzero subgroup H of G (this appears as an exercise in [10]). Jensen and Miller translated this question to commutative semigroups [9]. They defined an infinite commutative semigroup S to be *homomorphically finite* (HF for short) if and only if every proper homomorphic image of S is finite, and then proceeded to classify all HF commutative semigroups. Ralph Tucci defined an infinite commutative semigroup S to be *H-smaller* if and only if every proper homomorphic image of S has smaller cardinality than S. He showed that the H-smaller semigroups coincide with the HF semigroups [17].

Chew and Lawn defined a ring R with 1 (not assumed commutative) to be *residually finite* provided every proper homomorphic image of R is finite. They proved various results about such rings [1]. Levitz and Mott extended their results to rings without identity [12]. Unfortunately, this definition is not unique in the literature. Orzech and Ribes [16] defined an associative ring R to be residually finite if and only if for every nonzero $x \in R$, there is a two-sided ideal A of R such that $x \notin A$ and R/A is finite (this appears to be the standard definition).

Varadarajan [18] generalized this definition and called an R-module M residually finite if and only if for any $x \neq 0$ in M, there exists a submodule N of M (depending on x) such that $x \notin N$ and M/N is finite.

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In this paper, we study a variant of the notion considered in [18]. Keeping with Tucci's terminology, we define a module M over a ring R to be *H-smaller* (HS for short) if and only if M is infinite and $|M/N| < |M|$ for every nonzero submodule N of M. We begin by constructing nontrivial examples of HS modules over both Noetherian and non-Noetherian domains. In Section 3, we prove a structure theorem for HS modules (Theorem 3.3 is the principal result of this section). We then generalize Kaplansky's problem by replacing $\mathbb{Z}$ with an arbitrary infinite module over an arbitrary ring. We conclude the paper by applying our results to solve an open problem on Jónsson modules.

2 Examples

We begin by introducing several canonical examples of domains which are HS as modules over themselves. The verification is easy and is omitted.

Examples 2.1 The following domains are HS as modules over themselves.

- Infinite fields.
- The ring $\mathbb{Z}$ of integers.
- The ring $\mathbb{F}[x]$, where $\mathbb{F}$ is a finite field.

We now investigate the existence of uncountable HS modules over a Noetherian domain D which is not a field (the case where D is a field being trivial). To simplify terminology, we will call a domain D an *HS domain* if and only if D is not a field and D is HS as a module over itself. Before proceeding, we recall a few results from earlier papers.

Lemma 2.2 ([11] Lemma 2.1, Theorem 2.6) *The following hold.*

(i) *Suppose that D is a Noetherian integral domain that is not a finite field and let I be a proper ideal of D. If $|D| = \rho$ and $|D/I| = \kappa$, then $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$.*

(ii) *Conversely, let $(\rho, \kappa_1, \ldots, \kappa_n)$ be a finite sequence of cardinals, where each κ_i is a prime power or is infinite and $\kappa_i + \aleph_0 \leq \rho \leq \kappa_i^{\aleph_0}$ holds for all i. Then there is a principal ideal domain D (which is not a field) with exactly n maximal ideals $M_1, \ldots, M_n$ such that $|D| = \rho$ and $|D/M_i| = \kappa_i$ for all i.*

Proof (ii) when $n = 1$ (see [11] for further details). Let κ be either a prime power or infinite and suppose that $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$. Let F be a field of cardinality κ, and let $F[[t]]$ be the ring of formal power series over F in the variable t. The underlying set of $F[[t]]$ is the set of all functions from ω into F, whence $|F[[t]]| = \kappa^{\aleph_0}$. The quotient field of $F[[t]]$ is the field $F((t))$ of formal Laurent series in the variable t. There is a field K of cardinality ρ such that $F(t) \subseteq K \subseteq F((t))$ for any ρ satisfying $|F(t)| = \kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0} = |F((t))|$. Note that $F[[t]]$ is a discrete valuation ring (DVR) on $F((t))$, $K \subseteq F((t))$, and $F[[t]] \cap K$ is not a field (since t is not invertible). It follows that $F[[t]] \cap K$ is a DVR on K (whence also has cardinality ρ) with maximal ideal $M = (t) \cap K$. Clearly, F maps injectively into $(F[[t]] \cap K)/M$ and $(F[[t]] \cap K)/M$ maps injectively into $F[[t]]/(t) \cong F$. It follows that $|(F[[t]] \cap K)/M| = \kappa$ and the proof is complete. ∎

Lemma 2.3 ([14, Lemma 3]) *Let R be a ring and I be a finitely generated ideal of R. Then*

(i) *if R/I is finite, then R/I^n is finite for every positive integer n;*
(ii) *if R/I has infinite cardinality κ, then R/I^n has cardinality κ for every positive integer n.*

We now characterize the cardinals ρ for which there exists a Noetherian HS domain D of cardinality ρ.

Proposition 2.4 *Let ρ be an infinite cardinal. There exists a Noetherian HS domain D of cardinality ρ if and only if there exists a cardinal κ such that $1 < \kappa < \rho \leq \kappa^{\aleph_0}$.*

Proof Let ρ be an infinite cardinal. Suppose first that D is a Noetherian HS domain of cardinality ρ. Let I be a nonzero proper ideal of D. Then $1 < |D/I| := \kappa < \rho$. It follows from Lemma 2.2(i) that $\rho \leq \kappa^{\aleph_0}$. Conversely, suppose that κ satisfies $1 < \kappa < \rho \leq \kappa^{\aleph_0}$. By Lemma 2.2(ii), there is a DVR D with maximal ideal J such that $|D| = \rho$ and $|D/J| \leq \kappa$ (if κ is finite but not a power of a prime, simply note that $2 \leq \kappa$ and apply (ii)). It is well known that every nonzero ideal of D is a power of J. Thus it follows from Lemma 2.3 that $|D/I| < |D|$ for every nonzero ideal I of D. ■

Remark 2.5 It is a well known consequence of König's theorem that $\kappa < \kappa^{\operatorname{cf}\kappa}$ for every infinite κ ($\operatorname{cf}\kappa$ denotes the cofinality of κ). In particular, if κ has countable cofinality, then $\kappa < \kappa^{\aleph_0}$. It follows that there are Noetherian HS domains of arbitrarily large cardinality. Proposition 2.4 also shows that there are independence issues associated with cardinalities of Noetherian HS domains. For instance, it is undecidable in ZFC whether there exists a Noetherian HS domain D of size $\aleph_2$. Note that if CH fails, then $\aleph_2 \leq 2^{\aleph_0}$. Thus $1 < 2 < \aleph_2 \leq 2^{\aleph_0}$ and there exists a Noetherian HS domain of size $\aleph_2$ by Proposition 2.4. On the other hand, suppose that GCH holds. Note that if $\kappa < \aleph_2$, then $\kappa \leq \aleph_1$. Thus $\kappa^{\aleph_0} \leq (\aleph_1)^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \aleph_1 < \aleph_2$ and hence there can be no Noetherian HS domain of size $\aleph_2$.

We now explore the question of the existence of HS domains D that are not Noetherian. We will show that, unlike in the Noetherian case, such domains exist of every uncountable cardinality (it will follow from a later theorem that every countable HS domain is Noetherian). We begin with a simple lemma.

Lemma 2.6 *Let M be an infinite module over the ring R. Then M is HS over R if and only if $|M/(m)| < |M|$ for every nonzero $m \in M$.*

Proof If M is HS, then $|M/(m)| < |M|$ holds for every nonzero $m \in M$ by definition. Conversely, suppose $|M/(m)| < |M|$ for every nonzero $m \in M$. Let N be an arbitrary nonzero submodule of M and pick any nonzero $n \in N$. Now simply observe that $|M/N| \leq |M/(n)| < |M|$. ■

In what follows, κ will remain an arbitrary, but fixed, uncountable cardinal. Our construction will proceed roughly as follows. First we will show that there exists a torsion-free totally ordered abelian group G of cardinality κ such that the interval $(0, g) := \{x \in G : 0 < x < g\}$ has cardinality less than κ for every $g > 0$. We then consider the canonical valuation v on the quotient field K of the group ring $\mathbb{F}_2[G]$ into G. Let V be the valuation ring associated with v. We will show that V is an HS domain. We begin with the following lemma. We remark that the group constructed below appears explicitly in [5], though a different result is obtained there.

Lemma 2.7 *There exists a torsion-free totally ordered abelian group G of cardinality κ such that for every $g > 0$ in G, $(0, g) := \{x \in G : 0 < x < g\}$ has cardinality less than κ.*

Proof Let $G := \bigoplus_\kappa \mathbb{Z}$ denote the direct sum of κ copies of $\mathbb{Z}$. Clearly G is a torsion-free abelian group of cardinality κ. We will equip G with the reverse lexicographic order. The details follow. Each nonzero $(g_i) \in G$ has but finitely many nonzero coordinates. In other words, the set $\{i \in \kappa : g_i \neq 0\}$ is finite. Let j be the largest element of $\{i \in \kappa : g_i \neq 0\}$. We define (g_i) to be positive if and only if $g_j > 0$. Let P be the collection of positive elements of G. One checks easily that P is closed under addition and $\{P, \{0\}, -P\}$ forms a partition of G. Thus the order $<$ on G defined by $x < y$ if and only if $y - x \in P$ is total and respects addition. Let $g > 0$ be arbitrary. We show that $|(0, g)| < \kappa = |G|$. Let $g := (g_i)$ and let $j \in \kappa$ be the largest nonzero coordinate of g (hence $g_j > 0$). Now let $h := (h_i) \in (0, g)$ be arbitrary, and let j' be the largest nonzero coordinate of h. Since h is positive, $h_{j'} > 0$. But $g - h > 0$, and thus $j' \leq j$. It follows that the interval $(0, g)$ may be mapped injectively into $\bigoplus_{i \in j+1} \mathbb{Z}$. Since κ is a cardinal, κ is a limit ordinal, and hence $j + 1 \in \kappa$. Thus $|j + 1| < \kappa$, and it follows easily that $|\bigoplus_{i \in j+1} \mathbb{Z}| < \kappa$. Finally, we conclude that $|(0, g)| < \kappa$ and the proof is complete. ■

Theorem 2.8 *There exists an HS valuation domain V of cardinality κ.*

Proof Let G be a torsion-free totally ordered abelian group of cardinality κ such that $|[0, g]| < \kappa$ for every $g > 0$ in G (guaranteed by Lemma 2.7). Let S denote the subsemigroup of G consisting of the non-negative elements of G, and let $\mathbb{F}_2[S]$ denote the semigroup ring of S over $\mathbb{F}_2$. Note that every nonzero element of $D := \mathbb{F}_2[S]$ may be written uniquely in the form

$$x^{g_1} + x^{g_2} + \cdots + x^{g_n}, \tag{2.1}$$

where $g_1 < g_2 < \cdots < g_n$. Whenever we express an element of D as in (2.1), we will assume that $g_1 < g_2 < \cdots < g_n$.

We now define a map $v \colon D \to G \cup \{\infty\}$ by $v(x^{g_1} + x^{g_2} + \cdots + x^{g_n}) := g_1$ and $v(0) := \infty$ (recall that $g + \infty := \infty + g := \infty$ and $g < \infty$ for every $g \in G$). Let K be the quotient field of D. It follows (from [3, Proposition 18.1], for example) that v may be extended to a valuation on K by setting $v(a/b) = v(a) - v(b)$. We let $V = \{\alpha \in K : v(\alpha) \geq 0\}$ be the valuation ring of v. Recall that $u \in V$ is a unit if and only if $v(u) = 0$. Clearly V is not a field and V has cardinality κ. We now prove that V is HS.

Claim *If $\alpha \in V$ is nonzero, then $|V/(\alpha)| < |V| = \kappa$.*

Proof of Claim If α is a unit, the result is patent. Thus we assume α is a nonzero nonunit. Hence $v(\alpha) := g > 0$ and $g \in G$. Recall that $|[0,g]| < \kappa$. We now let $\mathcal{R}$ denote the elements of V whose exponents are contained in $[0,g]$. Formally, we have

$$\mathcal{R} := \left\{ \frac{x^{a_1} + x^{a_2} + \cdots + x^{a_n}}{x^{b_1} + x^{b_2} + \cdots + x^{b_m}} \in V : 0 \le a_i, b_j \le g \right\}.$$

Note that $|\mathcal{R}| < \kappa$. It suffices to show that $\mathcal{R}$ is a complete set of coset representatives for $V(\operatorname{mod} \alpha)$. Let $\beta \in V$ be arbitrary. Note that there exists $r \in \mathcal{R}$ such that $(\alpha) + \beta = (\alpha) + r$ if and only if there exists $r \in \mathcal{R}$ such that $\beta - r \in (\alpha)$ if and only if there exists $r \in \mathcal{R}$ such that $\frac{\beta - r}{\alpha} \in V$ if and only if there exists $r \in \mathcal{R}$ such that $v(\beta - r) \ge v(\alpha) = g$. If $\beta = 0$, choose $r := x^g$. Thus we assume that $\beta \ne 0$. Let $\beta := \frac{x^{g_1} + x^{g_2} + \cdots + x^{g_n}}{x^{h_1} + x^{h_2} + \cdots + x^{h_m}}$. Since $\beta \in V$, it follows by definition that $v(\beta) \ge 0$. Hence $g_1 \ge h_1$ and we may factor out x^{h_1} from the numerator and denominator. So without loss of generality, we may assume β has the form

$$\beta = \frac{x^{g_1} + x^{g_2} + \cdots + x^{g_n}}{1 + x^{h_1} + \cdots + x^{h_m}}, \quad 0 \le g_1 < g_2 < \cdots < g_n, 0 < h_1 < h_2 < \cdots < h_m.$$

If each g_i, h_j satisfies $0 \le g_i, h_j \le g$, then $\beta \in \mathcal{R}$, and we may choose $r := \beta$. Thus we assume that at least one g_i or h_j is larger than g. Note that $v(\beta) = g_1$. If $g_1 > g$, then we may choose $r = 0$. Thus we assume that $g_1 \le g$. Let i, $1 \le i \le n$ be largest such that $g_i \le g$ and let j, $0 \le j \le m$ be largest such that $h_j \le g$. Let $r := \frac{x^{g_1} + \cdots + x^{g_i}}{1 + \cdots + x^{h_j}}$. Note that as $v(r) = g_1$, we have $r \in V$. It follows that $r \in \mathcal{R}$. We will be done if we can show that $v(\beta - r) \ge g$. Simple algebra (obtaining a common denominator and cancelling) yields

$$\beta - r = \frac{(1 + \cdots + x^{h_j})(x^{g_{i+1}} + \cdots + x^{g_n}) - (x^{g_1} + \cdots + x^{g_i})(x^{h_{j+1}} + \cdots + x^{h_m})}{(1 + \cdots + x^{h_j})(1 + x^{h_1} + x^{h_2} + \cdots + x^{h_m})}.$$

Note that upon multiplying out in the numerator, each exponent is larger than g. Upon multiplying out in the denominator, one still has a constant term of $1 = x^0$. Thus $v(\beta - r) > g$ and the proof of the claim is complete. ■

It now follows from Lemma 2.6 that V is an HS domain. ■

Remark 2.9 One can extract even more information from the previous proof. By allowing g to vary over the nonnegative elements of G, it is possible to obtain intervals $[0,g]$ of *any* nonzero cardinality $\lambda < \kappa$. Thus there exist residue rings of V of finite cardinality as well as residue rings of cardinality λ for any infinite $\lambda < \kappa$. This contrasts sharply with the Noetherian case (recall Lemma 2.2(i)).

3 A Structure Theorem

Our objective in this section is to present some results on the structure of general HS modules. We begin with a lemma that collects some basic facts about these modules.

Lemma 3.1 *Let R be a ring, M an HS module over R, and let N be a nonzero submodule of M. Then the following hold:*

(i) $|N| = |M|$;
(ii) $|M| \leq |R|$;
(iii) *N is an HS module.*

Proof We assume that M is an HS module over the ring R and that N is a nonzero submodule of M.

(i) Suppose by way of contradiction that $|N| < |M|$. Since M is infinite, $|M/N| = |M|$, contradicting that M is HS.

(ii) If $|M| > |R|$, then choose any nonzero $m \in M$. Clearly $|(m)| < |M|$, and we have a contradiction to (i).

(iii) Let K be a nonzero submodule of N. Then N/K is a submodule of M/K. Since M is HS, it follows that $|N/K| \leq |M/K| < |M| = |N|$ (the last equality holds by (i)). ∎

Note that either (i) or (ii) above easily yields the following generalization of Kaplansky's problem (also observed by Tucci). If G is any infinite abelian group such that $|G/H| < |G|$ for every nonzero subgroup H of G, then $G \cong \mathbb{Z}$.

The following proposition will form the cornerstone of the proof of the main result of this section.

Proposition 3.2 *Suppose that M is an HS R-module. Then the following hold.*

(i) $\text{Ann}(m)$ *is a prime ideal of R for every nonzero $m \in M$.*
(ii) *The set $\{\text{Ann}(m) : m \in M\}$ is linearly ordered by inclusion.*
(iii) $\text{Ann}(M) = \text{Ann}(m)$ *for any nonzero $m \in M$.*
(iv) $\text{Ann}(M)$ *is a prime ideal of R.*

Proof We prove each of these in succession.

(i) Suppose by way of contradiction that there exists some nonzero $m \in M$ for which $\text{Ann}(m)$ is not prime. Then there exist $x, y \in R$ such that $xy \in \text{Ann}(m)$ but $x \notin \text{Ann}(m)$ and $y \notin \text{Ann}(m)$. It is easy to see that this implies $\text{Ann}(m) \subsetneq \text{Ann}(xm)$. Define $\varphi\colon (m) \to (xm)$ by $\varphi(rm) := rxm$. Let K be the kernel of this map. Since $\text{Ann}(m) \subsetneq \text{Ann}(xm)$, it follows that K is nonzero. By Lemma 3.1(iii), (m) is HS. Thus $|(m)/K| < |(m)|$. But as $(m)/K \cong (xm)$, we obtain $|(xm)| < |(m)|$ and we have a contradiction to Lemma 3.1(i).

(ii) Suppose by way of contradiction that there exist elements m and n in M with $\text{Ann}(m) = P$ and $\text{Ann}(n) = Q$, but $P \nsubseteq Q$ and $Q \nsubseteq P$. Let $p \in P - Q$ and $q \in Q - P$. Note that $pq \in \text{Ann}(m+n)$. Since $\text{Ann}(m+n)$ is prime (by (i)), we may assume that $p \in \text{Ann}(m+n)$. Thus $p(m+n) = 0$. However, since $p \in \text{Ann}(m)$, it follows that $pn = 0$. Hence $p \in \text{Ann}(n) = Q$, which is a contradiction.

(iii) Clearly, it suffices to show that $\text{Ann}(m) = \text{Ann}(n)$ for any nonzero $n, m \in M$. Again, we suppose not. Then by (ii), there exist nonzero $m, n \in M$ with $\text{Ann}(m) \subsetneq \text{Ann}(n)$. Hence the map $\varphi\colon (m) \to (n)$ defined by $\varphi(rm) := (rn)$ is well defined. As in (i), if K is the kernel of this map, then $K \neq \{0\}$. By Lemma 3.1(iii), (m) is HS. Thus $|(m)/K| < |(m)|$. But as $(m)/K \cong (n)$, we have $|(n)| < |(m)|$. This is a contradiction to Lemma 3.1(i).

(iv) This is immediate from (i) and (iii). ■

Thus by modding out the annihilator, there is no loss of generality in restricting our study to faithful modules over a domain. We now prove our main result.

Theorem 3.3 *Let D be a domain with quotient field K and let M be a faithful module over D. Consider the following conditions.*

(i) *D is HS as a module over itself.*
(ii) *$D \subseteq M \subseteq K$ (up to isomorphism).*
(iii) *There is a generating set S for M over D with $|S| < |D|$.*
(iv) $|M/D| < |D|$.

If M is an HS D-module, then conditions (i)–(iv) *hold. Conversely, if conditions* (i), (ii), *and* (iv) *hold, then M is HS over D.*

Proof Let D be a domain with quotient field K and suppose that M is a faithful HS module over D.

(i) Let m be an arbitrary nonzero element of M. By Proposition 3.2(iii), $\mathrm{Ann}(M) = \mathrm{Ann}(m)$. Since M is faithful, it follows that $\mathrm{Ann}(m) = \{0\}$ and thus M is torsion-free. This implies that $(m) \cong D$, and it follows from Lemma 3.1(iii) that D is HS as a module over itself. This establishes (i).

(ii) Since M is torsion-free, it suffices to show that M has rank one over D. Suppose by way of contradiction that $x, y \in M$ are distinct and linearly independent over D. Then clearly this implies that $D \oplus D$ is a submodule of M, whence $D \oplus D$ is an HS module by Lemma 3.1(iii). However, $|(D \oplus D)/D| = |D| = |D \oplus D|$, and this is a contradiction.

(iii) Let $m \in M$ be nonzero. Then $|M/(m)| < |M| = |D|$. Let S be a complete set of coset representatives for $M(\mathrm{mod}\ m)$. It is easy to check that $M = (S, m)$. Clearly $|S \cup \{m\}| < |D|$ and (iii) is established.

(iv) This follows easily from (ii) and the fact that M is HS.

Conversely, suppose that conditions (i), (ii), and (iv) are satisfied. We will show that M is HS. Suppose that N is a nonzero submodule of M. Note that by the isomorphism theorems, we have

$$(M/(N \cap D))/(D/(N \cap D)) \cong M/D. \tag{3.1}$$

Since N is nonzero and $N \subseteq K$, it follows that $N \cap D$ is a nonzero ideal of D. Since D is HS over itself, it follows that

$$|D/(N \cap D)| < |D|. \tag{3.2}$$

By (iii), $|M/D| < |D|$. This fact, along with (3.1) and (3.2), implies that

$$|M/(N \cap D)| < |D|.$$

Now simply note that $N \cap D \subseteq N$. Thus

$$|M/N| \leq |M/(N \cap D)| < |D| = |M|.$$ ■

We easily obtain the following corollary.

Corollary 3.4 *Suppose D is an uncountable principal ideal domain of cardinality ρ with exactly n maximal ideals $J_1, J_2, \ldots, J_n$. Suppose further that $|D/J_i| < |D|$ for $1 \leq i \leq n$. If K is the quotient field of D, then the HS modules over D are precisely (up to isomorphism) the D-modules lying between D and K (note that such domains D exist by Lemma* 2.2(ii)*).*

Proof We suppose that D is an uncountable principal ideal domain of cardinality ρ with exactly n maximal ideals $J_1, J_2, \ldots, J_n$ and $|D/J_i| < |D|$ for $1 \leq i \leq n$. Let K be the quotient field of D. We first show that D is HS as a module over itself. For each i, let $J_i = (p_i)$. If $x \in D$ is nonzero, then $x = up_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}$ for some unit u and some non-negative integers $k_1, k_2, \ldots, k_n$. Thus $(x) = (p_1^{k_1})(p_2^{k_2})\cdots(p_n^{k_n})$, and we have

$$D/(x) = D/(p_1^{k_1})(p_2^{k_2})\cdots(p_n^{k_n}) \cong D/(p_1)^{k_1} \oplus D/(p_2)^{k_2} \cdots \oplus D/(p_n)^{k_n}.$$

It now follows easily from Lemma 2.3 that $|D/(x)| < |D|$, and thus D is HS as a module over itself.

By Theorem 3.3, it suffices to show that $|K/D| < |D|$ to finish the proof. The following is well-known.

$$K/D \cong C(p_1^{\infty}) \oplus C(p_2^{\infty}) \oplus \cdots \oplus C(p_n^{\infty}),$$

where each $C(p_i^{\infty}) = \varinjlim D/(p_i)^k$. Lemma 2.3 now implies that $|K/D| < |D|$ and therefore $|M/D| < |D|$ for any D-module M with $D \subseteq M \subseteq K$. ■

In what follows, we will have plenty to say about what happens if D is countable.

4 A Generalization of Kaplansky's Problem

We begin by giving the canonical solution to Kaplansky's problem [10]. *Show that if G is an infinite abelian group with the property that G/H is finite for all nonzero subgroups H of G, then $G \cong \mathbb{Z}$.*

Solution Assume that G is as stated. Let $g \in G$ be nonzero. It is easy to see that $G = (g, X)$, where X is a complete set of coset representatives for $G/(g)$. Since $G/(g)$ is finite, it follows that G is finitely generated. Thus by the fundamental theorem of finitely generated abelian groups, G is a finite direct sum of cyclic groups. Since G is infinite, at least one summand must be isomorphic to $\mathbb{Z}$. There can be no other summands, lest $\mathbb{Z}$ be an infinite proper homomorphic image of G. Thus $G \cong \mathbb{Z}$. ■

We now replace $\mathbb{Z}$ with an arbitrary module M over an arbitrary ring R. We will use Jensen and Miller's terminology and call a module M *homomorphically finite* (HF for short) if and only if M is infinite but M/N is finite for all nonzero submodules N of M. We will now give a complete description of these modules, significantly generalizing Kaplansky's problem above. We first recall the following result of Chew and Lawn.

Lemma 4.1 ([1, Corollary 2.4]) *Let R be a commutative ring. Then every nonzero ideal I of R is of finite index in R if and only if every nonzero prime ideal of R is finitely generated and of finite index in R.*

Theorem 4.2 *Let D be a domain (which is not a field) with quotient field K and let M be a faithful module over D. Then M is HF if and only if the following hold.*

(i) *D is a one-dimensional Noetherian domain with all residue fields finite.*
(ii) $D \subseteq M \subseteq K$.
(iii) *M is finitely generated over D.*

Proof Let D be a domain (which is not a field) with quotient field K and let M be a faithful module over D. Assume first that M is HF. It follows from the proof of Theorem 3.3(i) that D is HF as a module over itself. Lemma 4.1 (along with Cohen's theorem) implies that D is Noetherian. If P is any nonzero prime ideal of D, then as D/P is finite, it follows that D/P is a field. Hence P is maximal. Thus (i) holds. Condition (ii) is the same as Theorem 3.3(ii). As for (iii), let $m \in M$ be nonzero. Then $M/(m)$ is finite. If S is a complete set of coset representatives for M mod (m), then as we have seen, $M = (S, m)$. Hence M is finitely generated.

Conversely, suppose that conditions (i)–(iii) above are satisfied. It follows from Lemma 4.1 that D is HF as a module over itself. Since M is finitely generated and $D \subseteq M \subseteq K$, it follows that there exists some nonzero $d \in D$ such that $dM \subseteq D$. But note that $M \cong dM$, and thus M is isomorphic to a submodule of D. It now follows from Lemma 3.1(iii) that M is HF. ■

Remark 4.3 Theorem 4.2 and Lemma 2.2(i) imply that if M is HF over D, then $|D| = |M| \leq 2^{\aleph_0}$. Note also that if D is a countably infinite domain and M is HS over D, then it follows from Lemma 3.1 that M is also countable. Thus M is HF and we obtain a description of the HS modules over an arbitrary countable ring. Recall, however, that if M is an HF module over D, D need not be countable (as $\mathbb{F}_2[[t]]$ witnesses).

5 An Application to Jónsson Modules

In the final section, we use the results of this paper to solve an open problem on Jónsson modules. We begin with a brief introduction to initiate the reader.

In universal algebra, an *algebra* is a pair $(X, \mathbf{F})$ consisting of a set X and a collection $\mathbf{F}$ of operations on X (there is no restriction placed on the arity of these operations). In case $\mathbf{F}$ is countable and all operations have finite arity, then $(X, \mathbf{F})$ is called a *Jónsson algebra* provided each proper subalgebra of X has smaller cardinality than X. Such algebras are of particular interest to set theorists. In set theory, a cardinal κ is said to be a *Jónsson cardinal* provided there is no Jónsson algebra of cardinality κ. Many papers have been written on this topic; we refer the reader to [2] for an excellent survey of these algebras.

In the early 1980's, Robert Gilmer and William Heinzer translated these notions to the realm of commutative algebra. In [8], they defined an infinite module M over a commutative ring R with identity to be a *Jónsson module* provided every proper submodule of M has smaller cardinality than M. They applied and extended their results in several subsequent papers [4, 6, 7]. Oman continued this study [13–15]. These papers contain most (if not all) of what is currently known on Jónsson modules, and we refer the reader to them for background.

It is not hard to see that there exist Jónsson modules of every infinite cardinality. Indeed, let F be an arbitrary infinite field and consider F as a module over itself. The only proper submodule of F is $\{0\}$, whence F is trivially a Jónsson module over itself. More generally, if R is a ring and J is a maximal ideal of infinite index, then R/J is a Jónsson module over R. Of course, such examples are not very interesting. Gilmer and Heinzer proved [8, Proposition 2.5] that if M is a Jónsson module over R, then the annihilator of M in R is a prime ideal of R. Thus there is no loss of generality in considering only faithful Jónsson modules over an integral domain. It is also not hard to show that if F is an infinite field, then the only Jónsson module over F is F itself. Thus we restrict our focus to faithful Jónsson modules over a domain D which is not a field.

Oman characterized the countable faithful Jónsson modules over a domain [14]. We recall this result below.

Proposition 5.1 ([14, theorem 2]) *Let D be a domain with quotient field K, and suppose that M is a countably infinite faithful D-module. Then M is a Jónsson module if and only if one of the following holds.*

(i) *D is a field and $M \cong D$.*
(ii) *There is a discrete valuation overring V of D with finite residue field such that M is a homomorphic image of K/V.*

The question of the existence of an uncountable faithful Jónsson module over a domain D that is not a field appears in [13–15]. We use the results of this paper to prove that such modules exist of every infinite cardinality. The following proposition gives a certain duality between a subclass of HS modules and Jónsson modules.

Proposition 5.2 *Let V be a valuation domain with quotient field K. If K/V is a Jónsson module over V, then V is an HS domain. Conversely, if V is an HS domain and if $|K/V| = |V|$, then K/V is a Jónsson module over V.*

Proof Let V be a valuation domain with quotient field K. Suppose first that K/V is a Jónsson module over V. Then by definition, K/V is infinite and hence V is not a field. Let $v \in V$ be nonzero. It is easy to check that $V/(v) \cong V/v \mod V$ (here V/v denotes the fractional ideal $\{x/v : x \in V\}$). As $V/v \mod V$ is a proper submodule of K/V and K/V is Jónsson, we obtain $|V/(v)| < |K/V| \leq |V|$. Lemma 2.6 implies that V is an HS domain. Conversely, suppose that V is an HS domain and $|K/V| = |V|$. We will show that K/V is a Jónsson module. Since K/V is uniserial (the submodules are linearly ordered by inclusion), it suffices to show that every cyclic submodule of K/V has smaller cardinality than K/V (since K/V is uniserial, every proper submodule of K/V is contained in a cyclic submodule). Consider a cyclic module $(\overline{x})$ where $x \in K$. If $x \in V$, then $(\overline{x}) = \{\overline{0}\}$ and we are done. Thus suppose $x \notin V$. Then $1/x \in V$. As above, $V/(1/x) \cong (\overline{x})$. Since V is HS, $|V/(1/x)| < |V| = |K/V|$. It follows that $|(\overline{x})| < |K/V|$ and the proof is complete. ■

Remark 5.3 We cannot dispense with the assumption that $|K/V| = |V|$ in the second part of Proposition 5.2. To see this, consider the HS domain $V := \mathbb{Q}[[t]]$, and let K be the quotient field of V. As we saw in the proof of Corollary 3.4, $K/V \cong \varinjlim V/(t)^n$. It follows from Lemma 2.3 that each $V/(t)^n$ is countable, and thus K/V is countable. Note that $\mathbb{Q}$ embeds properly into K/V via the map $q \mapsto V + \frac{q}{t}$, and thus K/V is not a Jónsson module over V.

We now prove the existence of Jónsson modules of any uncountable cardinality, solving an open problem from the literature.

Corollary 5.4 *Let κ be an uncountable cardinal. There exists a valuation domain V of cardinality κ with quotient field K such that K/V is a (faithful) Jónsson module over V of cardinality κ.*

Proof By Proposition 5.2, it suffices to show that the valuation domain V constructed in the proof of Theorem 2.8 satisfies $|K/V| = |V|$. Consider any distinct positive g and h in G. We claim that $1/x^g$ and $1/x^h$ are distinct mod V. Suppose not. Then $1/x^g - 1/x^h \in V$. But then $(x^h - x^g)/x^{g+h} \in V$. However, $v((x^h - x^g)/x^{g+h}) < 0$, and this is a contradiction. ■

We end the paper with an analog to Corollary 3.4.

Corollary 5.5 *Let V be a valuation domain with quotient field K and suppose that K/V is a Jónsson module over V. If M is a V-module such that $V \subseteq M \subsetneq K$, then M is HS over V.*

Proof Assume that V is a valuation domain with quotient field K and that K/V is a Jónsson module over V. By Proposition 5.2, V is an HS domain. Now suppose that $V \subseteq M \subsetneq K$. Then M/V is a proper submodule of K/V. Since K/V is Jónsson, $|M/V| < |K/V| \leq |V|$. It follows from Theorem 3.3 that M is HS over V. ■

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