J. Group Theory **12** (2009), 201–207
DOI 10.1515/JGT.2008.072

Journal of Group Theory

Endo-permutation modules arising from the action of a *p*-group on a defect zero block

Adam Salminen

(Communicated by G. Malle)

Abstract. Let p be an odd prime and let k be an algebraically closed field of characteristic p. Also, let G be a finite p'-group. By Maschke's theorem, kG is isomorphic to a product $\prod_{i=1}^{t} \mathrm{End}_k(V_i)$ as a k-algebra. Suppose that a p-subgroup P of $\mathrm{Aut}(G)$ stabilizes $\mathrm{End}_k(V_{i_0})$ for some i_0. Such a V_{i_0} will be an endo-permutation kP-module. Puig showed that the only modules that occur in this way are those whose image is torsion in the Dade group $D(P)$.

If G is any finite group and b is a defect zero block of kG, then $kGb \cong \mathrm{End}_k(L)$ for some L. If kGb is P-stable for some p-subgroup P of $\mathrm{Aut}(G)$ and $\mathrm{Br}_P(b) \neq 0$, then L will again be an endo-permutation kP-module. We show that if $p \geqslant 5$, then L is torsion in $D(P)$. This result depends on the classification of the finite simple groups.

1 Introduction

Let p be an odd prime, and let k be an algebraically closed field of characteristic p. Suppose that G is a finite p'-group. Then Maschke's theorem implies that we can write

$$kG \cong \prod_{i=1}^{t} M_{n_i}(k) \cong \prod_{i=1}^{t} \mathrm{End}_k(V_i).$$

Now suppose that P is a p-subgroup of $\mathrm{Aut}(G)$ that stabilizes $\mathrm{End}_k(V_{i_0})$ for some i_0; then V_{i_0} is an endo-permutation kP-module. It is natural to ask which endo-permutation kP-modules arise in this way.

Theorem 1.1 (Puig, [9]). *With the above set-up the modules V_i are torsion in the Dade group $D(P)$.*

The proof of the above result uses the fact that for every simple p'-group G, $\mathrm{Aut}(G)$ has p-rank 1, and the proof of this latter fact depends on the classification of finite simple groups. If we drop the assumption that G is a p'-group, then Maschke's theorem no longer applies. Write kG as a product $\prod_{i=1}^{t} B_i$ of indecomposable algebras. Each B_i has the form kGb_i for some primitive central idempotent b_i of kG, and both the algebras B_i and the idempotents b_i are called *blocks* of kG. A block kGb of kG is said to be a *defect zero* block if $kGb \cong \mathrm{End}_k(V)$ for some k-module V. As above, suppose that $kGb \cong \mathrm{End}_k(V)$ is a defect zero block of kG which is P-stable for some p-subgroup P of $\mathrm{Aut}(G)$. Also assume that $\mathrm{Br}_P(b) \neq 0$; then V is an endo-permutation kP-module. In this paper, we investigate which endo-permutation kP-modules appear in this way. It is expected that V is always torsion in $D(P)$. In this paper we show this is true for $p \geqslant 5$. Our first result is a consequence of a result of Carlson and Thévenaz [2].

Theorem 1.2. *Let p be odd. Assume the set-up and notation of the previous paragraph. If a non-torsion module V appears for some G, P and b, then we can find some G, P, V and b with $P \cong C_p \times C_p$ and V non-torsion in $D(P)$.*

Now that we have reduced to $P \cong C_p \times C_p$, we can apply the reduction results of [10], which depend on the classification of the finite simple groups. These results allow us to reduce to the cases when G is a central extension of $\mathrm{PSL}_{n+1}(q)$, $\mathrm{PSU}_{n+1}(q)$, or $D_4(q)$ with $p = 3$. A recent result of Kessar [6] takes care of the first two cases. So we are left with a single open case for $p = 3$. In particular we have the following.

Theorem 1.3. *Suppose that G is a finite group and $kGb \cong \mathrm{End}_k(V)$ is a defect zero block of kG which is P-stable for some p-subgroup P of* $\mathrm{Aut}(G)$. *Also assume that* $\mathrm{Br}_P(b) \neq 0$. *If $p \geqslant 5$, then V is torsion in $D(P)$. In particular, V is self-dual.*

The above result is a special case of the conjecture on the finiteness of the number of source algebra equivalence classes of nilpotent blocks, with defect group P, of finite groups for a fixed p-group P. A proof of this conjecture has been announced by Puig.

This paper is organized as follows. Section 2 recalls definitions and basic results on blocks and endo-permutation modules. Section 3 provides a proof of Theorem 1.2. Section 4 briefly recalls the results of [10] and contains a proof of Theorem 1.3.

2 Notation and preliminaries

Fix a prime p and an algebraically closed field k of characteristic p. Let G be a finite group, and let b be a block of kG. For a p-subgroup P of G, the *Brauer homomorphism* $\mathrm{Br}_P : (kG)^P \to kC_G(P)$ is defined by $\sum_{x \in G} \lambda_x x \mapsto \sum_{x \in C_G(P)} \lambda_x x$. It is easy to check that this map is a homomorphism. A *defect group* of a block b is defined to be a maximal p-subgroup P of G such that $\mathrm{Br}_P(b) \neq 0$. It is well known that any two defect groups are conjugate. A block is said to be a *defect zero* block if 1 is a defect group. It is also well known that b is a defect zero block if and only if $kGb \cong \mathrm{End}_k(L)$ for some k-module L.

We now recall the definitions and some facts about endo-permutation modules and the Dade group. For a more detailed discussion see [3] and [4] or [11]. Recall that if V is a kG-module, then the dual of $V^* = \mathrm{Hom}_k(V, k)$ is also a kG-module via the action $g \cdot \alpha(m) = \alpha(g^{-1}m)$ for $\alpha \in V^*$, $g \in G$ and $m \in V$. A kG-module V is a *permutation module* if it has a G-stable k-basis.

Definition 2.1 (Dade). Let P be a p-group. A kP-module V is said to be an *endo-permutation* module if $\mathrm{End}_k(V) \cong V \otimes V^*$ is a permutation kP-module.

The endo-permutation kP-modules which usually show up in the representation theory of finite groups are those which have a summand with vertex P. Such endo-permutation modules are said to be *capped*. Dade showed that V is capped if and only if k is a summand of $\mathrm{End}_k(V)$. We will follow the suggestion of Thévenaz in [11] and use 'endo-permutation' to mean 'capped endo-permutation' unless stated otherwise.

If V is an endo-permutation kP-module, then any two summands of vertex P are isomorphic. Such a summand is called a *cap* of V. Two endo-permutation kP-modules V and W are said to be equivalent if they have isomorphic caps. It is easy to see that this is an equivalence relation. We denote the class of an endo-permutation module V by $[V]$. The Dade group of a finite p-group P, denoted by $D(P)$, is the set of these equivalence relations with the group operation induced by the tensor product, that is, $[V] + [W] = [V \otimes W]$. The fact that this operation is well defined is the content of the following result.

Theorem 2.2 (Dade). *Let p be odd, and let P be a finite p-group. If V and W are endo-permutation kP-modules, then $V \otimes W$ is an endo-permutation kP-module. Moreover, the cap of $V \otimes W$ is isomorphic to a summand of $V_0 \otimes W_0$ where V_0 and W_0 are caps of V and W respectively.*

If V is a kP-module the *Brauer quotient* is defined by

$$V(P) = V^P \Big/ \Big(\sum_{Q<P} \mathrm{Tr}_Q^P(V^Q) \Big).$$

Suppose that $kGb \cong \mathrm{End}_k(V)$ is a defect zero block and that P is a p-group that acts on kGb. By [12, Corollary 21.4], V will inherit an action of P, and $V(P) \neq 0$ if and only if $\mathrm{Br}_P(b) \neq 0$. The following result which follows from Higman's criterion is contained in [11, Lemma 2.2].

Lemma 2.3. *Let V be an endo-permutation kP-module* (*not necessarily capped*). *The following are equivalent*:

(i) *V is capped*;

(ii) *the Brauer quotient* $(\mathrm{End}_k(V))(P)$ *is non-zero.*

Theorem 1.2 will be a consequence of the following result.

Theorem 2.4 (Carlson-Thévenaz [2, (13.1)]). *Let p be odd, and let P be a finite p-group. The map*

$$\prod_{R/Q} \mathrm{Defres}_{R/Q}^P : D(P) \to \prod_{R/Q} D(R/Q)$$

is injective, where R/Q runs over the set of all sections of P that are cyclic of order p or elementary abelian of rank 2.

The Defres maps are the composites of the ordinary restriction maps with the deflation maps, which we now describe. Let V be an endo-permutation kP-module. If Q is a normal subgroup of P, then it is easy to see that $(\mathrm{End}_k(V))^Q \cong \mathrm{End}_{kQ}(V)$ and that $(\mathrm{End}_k(V))^Q$ and $(\mathrm{End}_k(V))(Q)$ are acted on naturally by P/Q. By [3, Theorem 4.15] we have $\mathrm{End}_k(V)(Q) \cong \mathrm{End}_k(V_Q)$ for a unique endo-permutation $k(P/Q)$-module V_Q. In [3], Dade also showed that $D(C_p) = C_2$ for odd p. Combining this fact with Theorem 2.4, we have the following. If an endo-permutation kP-module V has a non-torsion image $[V]$ in $D(P)$, then there must be a section R/Q of P such that the image of $\mathrm{Defres}^P_{R/Q}(V)$ is non-torsion in $D(R/Q)$ and $R/Q \cong C_p \times C_p$. Another consequence of Theorem 2.4 is that if p is odd then any torsion element of $D(P)$ has order 2. It is also known that $-[V] = [V^*]$ for any $[V] \in D(P)$. Therefore $[V]$ is torsion in $D(P)$ if and only if V is self-dual.

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let G be a finite group. Suppose that b is a defect zero block. So $kGb \cong \mathrm{End}_k(V)$ for some k-module V. Further suppose that there is a p-group $P \leqslant \mathrm{Aut}(G)$ such that b is P-stable. Then [12, Lemma 28.1] shows that V is a P-module. The group algebra kG is a P-permutation module under the action of P. So the summand kGb of kG is also a permutation module by [3, (1.5)]. Therefore V is a (not necessarily capped) endo-permutation kP-module. If we also assume that $(\mathrm{End}_k(V))(P) \neq 0$, then V will be a capped endo-permutation module by Lemma 2.3.

We need the following result.

Lemma 3.1. *Let p be odd. Suppose that H is a finite group, and that kH has a defect zero block $kHc \cong \mathrm{End}_k(V)$ which is stable under an action of $Q \cong C_p \times C_p$ on H such that the V is non-torsion in $D(Q)$ under this action. Then the action of Q on H is faithful.*

Proof. Suppose that Q does not act faithfully on H. Let $R \leqslant Q$ be a subgroup which acts trivially on H. Then R clearly centralizes c and will therefore act trivially on V and on the dual V^* of V.

Since Q/R is a cyclic group, V is isomorphic to V^* as a kQ/R-modules. But this implies that V and V^* are isomorphic as kQ-modules, and self-dual Q-modules are torsion in $D(Q)$ from [2]. Therefore the action of Q on H must be faithful. □

We can now prove Theorem 1.2 which we restate for convenience.

Theorem 3.2. *Let p be odd. Suppose that there exist a finite group G and a defect zero block $kGb \cong \mathrm{End}_k(V)$ which is P-stable for some p-subgroup P of* $\mathrm{Aut}(G)$*. Also sup-*

pose that $(\mathrm{End}_k(V))(P) \neq 0$ *and the image of* V *in* $D(P)$ *is non-torsion. Then there exist* G, b, P *and* V *as above with* $P \cong C_p \times C_p$.

Proof. Assume that G, b, P and V are as above with $[V]$ non-torsion in $D(P)$ and

$$(\mathrm{End}_k(V))(P) \cong kGb(P) \neq 0.$$

As we mentioned above, there must be $Q \trianglelefteq R \leqslant P$ such that the image of $\mathrm{Defres}^P_{R/Q}(V)$ is non-torsion in $D(R/Q)$ and $R/Q \cong C_p \times C_p$. In the paragraph after Theorem 2.4, we noted that $\mathrm{Defres}^P_{R/Q}(V) \cong V_Q$ for some endo-permutation R/Q-module V_Q.

Let $\hat{G} = C_G(Q)$ and $\hat{b} = \mathrm{Br}_Q(b)$. Then $kGb(Q) = k\hat{G}\hat{b} \cong \mathrm{End}_k(V_Q)$. So $\hat{b}$ is a defect zero block of $\hat{G}$. The conjugation action of R/Q on $\hat{G}$ induces the same action on $k\hat{G}\hat{b}$ as the one induced by R/Q on V_Q. Since V_Q is non-torsion in $D(R/Q)$ this map must be injective by Lemma 3.1. This completes the proof of the theorem. □

4 Proof of Theorem 1.3

Assume that we can find a finite group G and a defect zero block $kGb \cong \mathrm{End}_k(V)$ of kG which is P-stable for a p-subgroup P of $\mathrm{Aut}(G)$. Also suppose that $\mathrm{Br}_P(b) \neq 0$. If V is not torsion in $D(P)$, then the results of the previous section allow us to assume that $P \cong C_p \times C_p$. This situation was considered in [10]. We recall two results from this paper.

Theorem 4.1 ([10]). *Suppose that* G *is a finite group such that* $C_p \times C_p \cong P \leqslant \mathrm{Aut}(G)$, *and that* b *is a* P*-stable defect zero block of* kG *such that* $\mathrm{Br}_P(b) \neq 0$. *Also, suppose that the source* V *of a simple* $k(G \rtimes P)b$*-module* M *is a finitely generated non-torsion endo-permutation* kP*-module. Then we can find* G, b, V *with the above properties where* G *is a central* p'*-extension of a simple group.*

A detailed proof of this result can be found in [10]. The main idea is to let N be a normal subgroup of G which is maximal with respect to being P-stable. We can find a P-stable block d of kN such that $bd \neq 0$. Applying Puig's algebra-theoretic version of Fong reduction reduces the problem to consideration of a central p'-extension of G/N. Our choice of N implies that G/N is a minimal normal subgroup of $G/N \rtimes P$. Therefore, G/N is a direct product of isomorphic simple groups. The direct product can be eliminated using the fact, which can be found in [1], that $\mathrm{Ten}^P_Q(\mathrm{End}_k(M)) \cong \mathrm{End}_k(\mathrm{Ten}^P_Q(M))$.

Now assume that G is a central p'-extension of a simple group. If b is a defect zero block of G, then the assumption that $\mathrm{Br}_P(b) \neq 0$ implies that $P \cap \mathrm{Inn}(G) = 1$, so that P can be detected in $\mathrm{Out}(G)$. Since $|\mathrm{Out}(G)| \leqslant 2$ for all sporadic groups, they cannot provide examples in the above theorem. Moreover $|\mathrm{Out}(G)| = 2$ for all alternating groups except A_6 and $|\mathrm{Out}(A_6)| = 4$, and so these groups fail to provide examples. This leaves the finite simple groups of Lie type. Looking at the structure of $\mathrm{Out}(G)$ reduces the problem to $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$ (with the restrictions listed below) or $p = 3$

and $D_4(p)$ (with the restrictions listed below), $E_6(q)$ or ${}^2E_6(q)$. This only involves looking at the structure of $\mathrm{Out}(G)$. In fact we have the more restrictive condition that $P \cap \mathrm{Inn}(G) = 1$ with $P \subseteq \mathrm{Aut}(G)$, and from this the groups E_6 and ${}^2E_6$ can also be eliminated, and we have the following result whose detailed proof can be found in [10].

Theorem 4.2 ([10]). *Assume that p is odd. Suppose that $P = C_p \times C_p \leqslant \mathrm{Aut}(G)$ where G is a finite group. Suppose that b is a P-stable defect zero block of kG such that $\mathrm{Br}_P(b) \neq 0$. Finally, suppose that the source V of a simple $k(G \rtimes P)b$-module M is a finitely generated endo-permutation kP-module whose image is non-torsion in $D(P)$. Then we can find G, b, V such that one of the following holds.*

(i) (a) *G is a central p'-extension of $A_n(q) = \mathrm{PSL}_{n+1}(q)$ with $p \mid (n+1, q-1, f)$ where $q = r^f$; or*
  (b) *G is a central p'-extension of ${}^2A_n(q) = \mathrm{PSU}_{n+1}(q)$ with $p \mid (n+1, q+1, f)$ where $q = r^f$; or*

(ii) *$p = 3$, $q = r^f$ and G is a central extension of $D_4(q)$ with $3 | f$.*

The following result of Kessar takes care of the cases of A and ${}^2A$ above.

Theorem 4.3 ([6, Theorem 1.2]). *Let p be odd, and let H be a finite group with a normal subgroup N such that H/N is elementary abelian of order p^2. Suppose that N is a quasi-simple group, with $Z(N)$ a p'-group and such that $N/Z(N)$ is isomorphic to $\mathrm{PSL}_n(q)$ or to $\mathrm{PSU}_n(q)$ where q is a prime power that is not divisible by p. Suppose that b is an H-stable block of kN which is of defect zero. Let U be a simple kHb-module and let (P, W) be a vertex source pair for U. Then $[W]$ has order at most 2 in $D(P)$.*

We can now prove the following result, stated earlier as Theorem 1.3.

Theorem 4.4. *Let $p > 3$ be a prime. Suppose that G is a finite group and*

$$kGb \cong \mathrm{End}_k(V)$$

is a defect zero block of kG which is P-stable for some p-subgroup P of $\mathrm{Aut}(G)$. Also assume that $\mathrm{Br}_P(b) \neq 0$. Then V is torsion in $D(P)$.

Proof. Let G, b, V and P be as above. Theorem 1.2 allows us to assume that $P \cong C_p \times C_p$. Then applying Theorem 4.2 we may assume that G is a central p'-extension of $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$ for some prime power q which is not divisible by p. Letting $N = G$ and $H = G \rtimes P$ we can apply Theorem 4.3 and conclude that V must be torsion. □

References

[1] S. Bouc and J. Thévenaz. The group of endo-permutation modules. *Invent. Math.* **139** (2000), 275–349.

[2] J. F. Carlson and J. Thévenaz. The classification of torsion endo-trivial modules. *Ann. of Math.* (2) **162** (2005), 823–883.
[3] E. C. Dade. Endo-permutation modules over p-groups I. *Ann. of Math.* (2) **107** (1978), 459–494.
[4] E. C. Dade. Endo-permutation modules over p-groups II. *Ann. of Math.* (2) **108** (1978), 317–346.
[5] D. Gorenstein, R. Lyons and R. M. Solomon. *The classification of the finite simple groups*, no. 3 (American Mathematical Society, 1998).
[6] R. Kessar. On duality inducing automorphisms and sources of simple modules in classical groups. (Preprint.)
[7] N. Mazza. Endo-permutation modules as sources of simple modules. *J. Group Theory* **6** (2003), 477–497.
[8] H. Nagao and Y. Tsushima. *Representations of finite groups* (Academic Press, 1989).
[9] L. Puig. Notes sur les algèbres de Dade. Unpublished manuscript (1988).
[10] A. Salminen. On the sources of simple modules in nilpotent blocks. *J. Algebra* **319** (2008), 4559–4574.
[11] J. Thévenaz. Endo-permutation modules, a guided tour. In *Group representation theory* (EPFL Press, Lausanne, 2007), pp. 115–147.
[12] J. Thévenaz. *G-algebras and modular representation theory* (Clarendon Press, 1995).

Received 11 April, 2008; revised 30 April, 2008

Adam Salminen, Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, IN 47722, U.S.A.
E-mail: as341@evansville.edu