Endo-permutation modules arising from the action of a $p$-group on a defect zero block

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Abstract. Let $p$ be an odd prime and let $k$ be an algebraically closed field of characteristic $p$. Also, let $G$ be a finite $p'$-group. By Maschke’s theorem, $kG$ is isomorphic to a product $\prod_{i=1}^{t} \operatorname{End}_k(V_i)$ as a $k$-algebra. Suppose that a $p$-subgroup $P$ of $\operatorname{Aut}(G)$ stabilizes $\operatorname{End}_k(V_{i_0})$ for some $i_0$. Such a $V_{i_0}$ will be an endo-permutation $kP$-module. Puig showed that the only modules that occur in this way are those whose image is torsion in the Dade group $D(P)$.

If $G$ is any finite group and $b$ is a defect zero block of $kG$, then $kGb \cong \operatorname{End}_k(L)$ for some $L$. If $kGb$ is $P$-stable for some $p$-subgroup $P$ of $\operatorname{Aut}(G)$ and $\operatorname{Br}_P(b) \neq 0$, then $L$ will again be an endo-permutation $kP$-module. We show that if $p \neq 5$, then $L$ is torsion in $D(P)$. This result depends on the classification of the finite simple groups.

1 Introduction

Let $p$ be an odd prime, and let $k$ be an algebraically closed field of characteristic $p$. Suppose that $G$ is a finite $p'$-group. Then Maschke’s theorem implies that we can write

$$kG \cong \prod_{i=1}^{t} M_{n_i}(k) \cong \prod_{i=1}^{t} \operatorname{End}_k(V_i).$$

Now suppose that $P$ is a $p$-subgroup of $\operatorname{Aut}(G)$ that stabilizes $\operatorname{End}_k(V_{i_0})$ for some $i_0$; then $V_{i_0}$ is an endo-permutation $kP$-module. It is natural to ask which endo-permutation $kP$-modules arise in this way.

Theorem 1.1 (Puig, [9]). With the above set-up the modules $V_i$ are torsion in the Dade group $D(P)$.

The proof of the above result uses the fact that for every simple $p'$-group $G$, $\operatorname{Aut}(G)$ has $p$-rank 1, and the proof of this latter fact depends on the classification of finite simple groups. If we drop the assumption that $G$ is a $p'$-group, then Maschke’s theorem no longer applies. Write $kG$ as a product $\prod_{i=1}^{t} B_i$ of indecompos-
able algebras. Each $B_i$ has the form $kGb_i$ for some primitive central idempotent $b_i$ of $kG$, and both the algebras $B_i$ and the idempotents $b_i$ are called blocks of $kG$. A block $kGb$ of $kG$ is said to be a defect zero block if $kGb \cong \text{End}_k(V)$ for some $k$-module $V$. As above, suppose that $kGb \cong \text{End}_k(V)$ is a defect zero block of $kG$ which is $P$-stable for some $p$-subgroup $P$ of $\text{Aut}(G)$. Also assume that $\text{Br}_P(b) \neq 0$; then $V$ is an endo-permutation $kP$-module. In this paper, we investigate which endo-permutation $kP$-modules appear in this way. It is expected that $V$ is always torsion in $D(P)$. In this paper we show this is true for $p \geq 5$. Our first result is a consequence of a result of Carlson and Thévenaz [2].

**Theorem 1.2.** Let $p$ be odd. Assume the set-up and notation of the previous paragraph. If a non-torsion module $V$ appears for some $G$, $P$ and $b$, then we can find some $G$, $P$, $V$ and $b$ with $P \cong C_p \times C_p$ and $V$ non-torsion in $D(P)$.

Now that we have reduced to $P \cong C_p \times C_p$, we can apply the reduction results of [10], which depend on the classification of the finite simple groups. These results allow us to reduce to the cases when $G$ is a central extension of $\text{PSL}_{n+1}(q)$, $\text{PSU}_{n+1}(q)$, or $D_4(q)$ with $p = 3$. A recent result of Kessar [6] takes care of the first two cases. So we are left with a single open case for $p = 3$. In particular we have the following.

**Theorem 1.3.** Suppose that $G$ is a finite group and $kGb \cong \text{End}_k(V)$ is a defect zero block of $kG$ which is $P$-stable for some $p$-subgroup $P$ of $\text{Aut}(G)$. Also assume that $\text{Br}_P(b) \neq 0$. If $p \geq 5$, then $V$ is torsion in $D(P)$. In particular, $V$ is self-dual.

The above result is a special case of the conjecture on the finiteness of the number of source algebra equivalence classes of nilpotent blocks, with defect group $P$, of finite groups for a fixed $p$-group $P$. A proof of this conjecture has been announced by Puig.

This paper is organized as follows. Section 2 recalls definitions and basic results on blocks and endo-permutation modules. Section 3 provides a proof of Theorem 1.2. Section 4 briefly recalls the results of [10] and contains a proof of Theorem 1.3.

## 2 Notation and preliminaries

Fix a prime $p$ and an algebraically closed field $k$ of characteristic $p$. Let $G$ be a finite group, and let $b$ be a block of $kG$. For a $p$-subgroup $P$ of $G$, the *Brauer homomorphism* $\text{Br}_P: (kG)^P \to kC_G(P)$ is defined by $\sum_{x \in G} \lambda_{x\cdot x} \mapsto \sum_{x \in C_G(P)} \lambda_{x\cdot x}$. It is easy to check that this map is a homomorphism. A defect group of a block $b$ is defined to be a maximal $p$-subgroup $P$ of $G$ such that $\text{Br}_P(b) \neq 0$. It is well known that any two defect groups are conjugate. A block is said to be a defect zero block if 1 is a defect group. It is also well known that $b$ is a defect zero block if and only if $kGb \cong \text{End}_k(L)$ for some $k$-module $L$.

We now recall the definitions and some facts about endo-permutation modules and the Dade group. For a more detailed discussion see [3] and [4] or [11]. Recall that if $V$ is a $kG$-module, then the dual of $V^* = \text{Hom}_k(V, k)$ is also a $kG$-module via the action $g \cdot \alpha(m) = \alpha(g^{-1}m)$ for $\alpha \in V^*$, $g \in G$ and $m \in V$. A $kG$-module $V$ is a permutation module if it has a $G$-stable $k$-basis.
Definition 2.1 (Dade). Let \( P \) be a \( p \)-group. A \( kP \)-module \( V \) is said to be an endo-permutation module if \( \text{End}_k(V) \cong V \otimes V^* \) is a permutation \( kP \)-module.

The endo-permutation \( kP \)-modules which usually show up in the representation theory of finite groups are those which have a summand with vertex \( P \). Such endo-permutation modules are said to be capped. Dade showed that \( V \) is capped if and only if \( k \) is a summand of \( \text{End}_k(V) \). We will follow the suggestion of Thévenaz in [11] and use ‘endo-permutation’ to mean ‘capped endo-permutation’ unless stated otherwise.

If \( V \) is an endo-permutation \( kP \)-module, then any two summands of vertex \( P \) are isomorphic. Such a summand is called a cap of \( V \). Two endo-permutation \( kP \)-modules \( V \) and \( W \) are said to be equivalent if they have isomorphic caps. It is easy to see that this is an equivalence relation. We denote the class of an endo-permutation module \( V \) by \( [V]_C \). The Dade group of a finite \( p \)-group \( P \), denoted by \( D(P) \), is the set of these equivalence relations with the group operation induced by the tensor product, that is, \([V] + [W] = [V \otimes W]\). The fact that this operation is well defined is the content of the following result.

Theorem 2.2 (Dade). Let \( p \) be odd, and let \( P \) be a finite \( p \)-group. If \( V \) and \( W \) are endo-permutation \( kP \)-modules, then \( V \otimes W \) is an endo-permutation \( kP \)-module. Moreover, the cap of \( V \otimes W \) is isomorphic to a summand of \( V_0 \otimes W_0 \) where \( V_0 \) and \( W_0 \) are caps of \( V \) and \( W \) respectively.

If \( V \) is a \( kP \)-module the Brauer quotient is defined by

\[
V(P) = V^P / \left( \sum_{Q < P} \text{Tr}_Q^P (V_Q) \right).
\]

Suppose that \( kGb \cong \text{End}_k(V) \) is a defect zero block and that \( P \) is a \( p \)-group that acts on \( kGb \). By [12, Corollary 21.4], \( V \) will inherit an action of \( P \), and \( V(P) \neq 0 \) if and only if \( \text{Br}_P(b) \neq 0 \). The following result which follows from Higman’s criterion is contained in [11, Lemma 2.2].

Lemma 2.3. Let \( V \) be an endo-permutation \( kP \)-module (not necessarily capped). The following are equivalent:

(i) \( V \) is capped;

(ii) the Brauer quotient \( (\text{End}_k(V))(P) \) is non-zero.

Theorem 1.2 will be a consequence of the following result.

Theorem 2.4 (Carlson-Thévenaz [2, (13.1)]). Let \( p \) be odd, and let \( P \) be a finite \( p \)-group. The map

\[
\prod_{R/Q} \text{Defres}_R^P \circ D(P) : D(P) \to \prod_{R/Q} D(R/Q)
\]
is injective, where \( R/Q \) runs over the set of all sections of \( P \) that are cyclic of order \( p \) or elementary abelian of rank 2.

The Defres maps are the composites of the ordinary restriction maps with the deflation maps, which we now describe. Let \( V \) be an endo-permutation \( kP \)-module. If \( Q \) is a normal subgroup of \( P \), then it is easy to see that \( \text{End}_k(V)^Q \cong \text{End}_k(V) \) and that \( \text{End}_k(V)(Q) \) and \( \text{End}_k(V)(Q) \) are acted on naturally by \( P/Q \). By [3, Theorem 4.15] we have \( \text{End}_k(V)(Q) \cong \text{End}_k(V_Q) \) for a unique endo-permutation \( k(P/Q) \)-module \( V_Q \). In [3], Dade also showed that \( D(C_p) = C_2 \) for odd \( p \). Combining this fact with Theorem 2.4, we have the following. If an endo-permutation \( kP \)-module \( V \) has a non-torsion image \( \frac{V}{C_3} \) in \( D(P) \), then there must be a section \( R/Q \) of \( P \) such that the image of \( \text{Defres}_{P/Q}(V) \) is non-torsion in \( D(R/Q) \) and \( R/Q \cong C_p \times C_p \). Another consequence of Theorem 2.4 is that if \( p \) is odd then any torsion element of \( D(P) \) has order 2. It is also known that \( [V] = [V^*] \) for any \( [V] \in D(P) \). Therefore \( [V] \) is torsion in \( D(P) \) if and only if \( V \) is self-dual.

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let \( G \) be a finite group. Suppose that \( b \) is a defect zero block. So \( kGb \cong \text{End}_k(V) \) for some \( k \)-module \( V \). Further suppose that there is a \( p \)-group \( P \leq \text{Aut}(G) \) such that \( b \) is \( P \)-stable. Then [12, Lemma 28.1] shows that \( V \) is a \( P \)-module. The group algebra \( kG \) is a \( P \)-permutation module under the action of \( P \). So the summand \( kGb \) of \( kG \) is also a permutation module by [3, (1.5)]. Therefore \( V \) is a (not necessarily capped) endo-permutation \( kP \)-module. If we also assume that \( \text{End}_k(V)(P) \neq 0 \), then \( V \) will be a capped endo-permutation module by Lemma 2.3.

We need the following result.

**Lemma 3.1.** Let \( p \) be odd. Suppose that \( H \) is a finite group, and that \( kH \) has a defect zero block \( kHc \cong \text{End}_k(V) \) which is stable under an action of \( Q \cong C_p \times C_p \) on \( H \) such that the \( V \) is non-torsion in \( D(Q) \) under this action. Then the action of \( Q \) on \( H \) is faithful.

**Proof.** Suppose that \( Q \) does not act faithfully on \( H \). Let \( R \leq Q \) be a subgroup which acts trivially on \( H \). Then \( R \) clearly centralizes \( c \) and will therefore act trivially on \( V \) and on the dual \( V^* \) of \( V \).

Since \( Q/R \) is a cyclic group, \( V \) is isomorphic to \( V^* \) as a \( kQ/R \)-modules. But this implies that \( V \) and \( V^* \) are isomorphic as \( kQ \)-modules, and self-dual \( Q \)-modules are torsion in \( D(Q) \) from [2]. Therefore the action of \( Q \) on \( H \) must be faithful. \( \Box \)

We can now prove Theorem 1.2 which we restate for convenience.

**Theorem 3.2.** Let \( p \) be odd. Suppose that there exist a finite group \( G \) and a defect zero block \( kGb \cong \text{End}_k(V) \) which is \( P \)-stable for some \( p \)-subgroup \( P \) of \( \text{Aut}(G) \). Also sup-
pose that \((\text{End}_k(V))(P) \neq 0\) and the image of \(V\) in \(D(P)\) is non-torsion. Then there exist \(G, b, P\) and \(V\) as above with \(P \cong C_p \times C_p\).

**Proof.** Assume that \(G, b, P\) and \(V\) are as above with \([V]\) non-torsion in \(D(P)\) and

\[
(\text{End}_k(V))(P) \cong kGb(P) \neq 0.
\]

As we mentioned above, there must be \(Q \subseteq R \leq P\) such that the image of \(\text{Defres}_{R/Q}(V)\) is non-torsion in \(D(R/Q)\) and \(R/Q \cong C_p \times C_p\). In the paragraph after Theorem 2.4, we noted that \(\text{Defres}_{R/Q}(V) \cong V_Q\) for some endo-permutation \(R/Q\)-module \(V_Q\).

Let \(G = C_G(Q)\) and \(\hat{b} = \text{Br}_Q(b)\). Then \(kGb(Q) = k\hat{G}\hat{b} \cong \text{End}_k(V_Q)\). So \(\hat{b}\) is a defect zero block of \(\hat{G}\). The conjugation action of \(R/Q\) on \(\hat{G}\) induces the same action on \(k\hat{G}\hat{b}\) as the one induced by \(R/Q\) on \(V_Q\). Since \(V_Q\) is non-torsion in \(D(R/Q)\) this map must be injective by Lemma 3.1. This completes the proof of the theorem. \(\square\)

### 4 Proof of Theorem 1.3

Assume that we can find a finite group \(G\) and a defect zero block \(kGb \cong \text{End}_k(V)\) of \(kG\) which is \(P\)-stable for a \(p\)-subgroup \(P\) of \(\text{Aut}(G)\). Also suppose that \(\text{Br}_P(b) \neq 0\). If \(V\) is not torsion in \(D(P)\), then the results of the previous section allow us to assume that \(P \cong C_p \times C_p\). This situation was considered in [10]. We recall two results from this paper.

**Theorem 4.1 ([10]).** Suppose that \(G\) is a finite group such that \(C_p \times C_p \cong P \subseteq \text{Aut}(G)\), and that \(b\) is a \(P\)-stable defect zero block of \(kG\) such that \(\text{Br}_P(b) \neq 0\). Also, suppose that the source \(V\) of a simple \(k(G \rtimes P)b\)-module \(M\) is a finitely generated non-torsion endo-permutation \(kP\)-module. Then we can find \(G, b, V\) with the above properties where \(G\) is a central \(p'\)-extension of a simple group.

A detailed proof of this result can be found in [10]. The main idea is to let \(N\) be a normal subgroup of \(G\) which is maximal with respect to being \(P\)-stable. We can find a \(P\)-stable block \(d\) of \(kN\) such that \(bd \neq 0\). Applying Puig’s algebra-theoretic version of Fong reduction reduces the problem to consideration of a central \(p'\)-extension of \(G/N\). Our choice of \(N\) implies that \(G/N\) is a minimal normal subgroup of \(G/N \rtimes P\). Therefore, \(G/N\) is a direct product of isomorphic simple groups. The direct product can be eliminated using the fact, which can be found in [1], that \(\text{Ten}_P^0(\text{End}_k(M)) \cong \text{End}_k(\text{Ten}_P^0(M))\).

Now assume that \(G\) is a central \(p'\)-extension of a simple group. If \(b\) is a defect zero block of \(G\), then the assumption that \(\text{Br}_P(b) \neq 0\) implies that \(P \cap \text{Inn}(G) = 1\), so that \(P\) can be detected in \(\text{Out}(G)\). Since \(|\text{Out}(G)| \leq 2\) for all sporadic groups, they cannot provide examples in the above theorem. Moreover \(|\text{Out}(G)| = 2\) for all alternating groups except \(A_6\) and \(|\text{Out}(A_6)| = 4\), and so these groups fail to provide examples. This leaves the finite simple groups of Lie type. Looking at the structure of \(\text{Out}(G)\) reduces the problem to \(\text{PSL}_n(q), \text{PSU}_n(q)\) (with the restrictions listed below) or \(p = 3\).
and $D_4(p)$ (with the restrictions listed below), $E_6(q)$ or $^2E_6(q)$. This only involves looking at the structure of $\text{Out}(G)$. In fact we have the more restrictive condition that $P \cap \text{Inn}(G) = 1$ with $P \subseteq \text{Aut}(G)$, and from this the groups $E_6$ and $^2E_6$ can also be eliminated, and we have the following result whose detailed proof can be found in [10].

**Theorem 4.2** ([10]). Assume that $p$ is odd. Suppose that $P = C_p \times C_p \leq \text{Aut}(G)$ where $G$ is a finite group. Suppose that $b$ is a $P$-stable defect zero block of $kG$ such that $\text{Br}_P(b) \neq 0$. Finally, suppose that the source $V$ of a simple $k(G \rtimes P)$-$b$-module $M$ is a finitely generated endo-permutation $kP$-module whose image is non-torsion in $D(P)$. Then we can find $G$, $b$, $V$ such that one of the following holds.

(i) (a) $G$ is a central $p'$-extension of $A_n(q) = \text{PSL}_{n+1}(q)$ with $p \mid (n+1,q-1,f)$ where $q = r^f$; or

(b) $G$ is a central $p'$-extension of $^2A_n(q) = \text{PSU}_{n+1}(q)$ with $p \mid (n+1,q+1,f)$ where $q = r^f$; or

(ii) $p = 3$, $q = r^f$ and $G$ is a central extension of $D_4(q)$ with $3 \mid f$.

The following result of Kessar takes care of the cases of $A$ and $^2A$ above.

**Theorem 4.3** ([6, Theorem 1.2]). Let $p$ be odd, and let $H$ be a finite group with a normal subgroup $N$ such that $H/N$ is elementary abelian of order $p^2$. Suppose that $N$ is a quasi-simple group, with $Z(N)$ a $p'$-group and such that $N/Z(N)$ is isomorphic to $\text{PSL}_n(q)$ or to $\text{PSU}_n(q)$ where $q$ is a prime power that is not divisible by $p$. Suppose that $b$ is an $H$-stable block of $kN$ which is of defect zero. Let $U$ be a simple $kHb$-module and let $(P, W)$ be a vertex source pair for $U$. Then $[W]$ has order at most 2 in $D(P)$.

We can now prove the following result, stated earlier as Theorem 1.3.

**Theorem 4.4.** Let $p > 3$ be a prime. Suppose that $G$ is a finite group and

$$kGb \cong \text{End}_k(V)$$

is a defect zero block of $kG$ which is $P$-stable for some $p$-subgroup $P$ of $\text{Aut}(G)$. Also assume that $\text{Br}_P(b) \neq 0$. Then $V$ is torsion in $D(P)$.

**Proof.** Let $G$, $b$, $V$ and $P$ be as above. Theorem 1.2 allows us to assume that $P \cong C_p \times C_p$. Then applying Theorem 4.2 we may assume that $G$ is a central $p'$-extension of $\text{PSL}_n(q)$ or $\text{PSU}_n(q)$ for some prime power $q$ which is not divisible by $p$. Letting $N = G$ and $H = G \rtimes P$ we can apply Theorem 4.3 and conclude that $V$ must be torsion.  

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