Finite subnormal coverings of certain solvable groups

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Dedicated to our teacher Luise-Charlotte Kappe

I. Introduction

A group is said to have a finite covering by subgroups if it is the set theoretic union of finitely many subgroups. A theorem of B. H. Neumann [11] characterizes groups with finite coverings by proper subgroups as precisely those groups with finite non-cyclic homomorphic images. R. Baer (see [13, Theorem 4.16]) proved that a group has a finite covering by abelian subgroups if and only if it is central-by-finite. Refinements of the above two results were given in [1] where finite coverings by normal subgroups were investigated. Specifically, the following theorems were established:
Theorem A A group $G$ has a finite covering by proper normal subgroups if and only if $G$ has an elementary abelian $p$-group of rank 2 as a homomorphic image for some prime $p$.

Theorem B A group $G$ has a finite covering by normal abelian subgroups if and only if $G$ is a central-by-finite 2-Engel group.

A natural question to ask is whether generalizations of Theorems A and B can be found by considering finite coverings by subnormal subgroups. Since it is clear that a group has a finite covering by proper subnormal subgroups if and only if it has a finite covering by proper normal subgroups, our focus will be on generalizations of Theorem B. For example, if a group $G$ is a set theoretic union of subnormal abelian subgroups of defect at most 2 (i.e. $G = \bigcup H_i$ with $H_i \triangleleft H_i^G \triangleleft G$), then it is straightforward to show that $G$ is a 3-Engel group (see Lemma 3 below). If this covering by subnormal abelian subgroups is finite then by Baer’s result the group is central-by-finite. On the other hand, if $G$ is a 3-Engel group, a result of Kappe and Kappe [5] states that the normal closure of each element of $G$ is 2-Engel. It follows that $Z(G)g^G$ is 2-Engel for all $g \in G$ where $Z(G)$ is the center of $G$. If $G$ is central-by-finite then the subgroups $Z(G)h_i^G$ form a finite covering of $G$ by normal 2-Engel subgroups where the $h_i$ form a transversal of $Z(G)$ in $G$. By Theorem B each of these 2-Engel groups has a finite covering by normal abelian subgroups. Hence we have the following result:

Theorem 1 A group $G$ has a finite covering by subnormal abelian subgroups of defect at most 2 if and only if $G$ is a central-by-finite 3-Engel group.

Note that the sufficiency of Theorem 1 follows from the fact that a group is 3-Engel exactly when it is covered by a system (not necessarily finite) of normal 2-Engel subgroups. These normal 2-Engel subgroups are in turn covered by a system of normal abelian subgroups. We introduce iterated Levi-properties which characterize those groups covered by a system of subnormal subgroups of bounded defect with a given property.
Levi-properties were introduced by L.-C. Kappe in [3] and are modeled after those groups first classified by F. W. Levi in which conjugates commute. Given a group theoretic class $\mathcal{X}$, a group is in the derived class $L(\mathcal{X})$ if the normal closure of each element in the group is an $\mathcal{X}$-group. The property of being in the class $L(\mathcal{X})$ is called the Levi-property generated by $\mathcal{X}$. Let $L_1(\mathcal{X}) = L(\mathcal{X})$. For $n > 1$, we define the $n$-th iterated Levi-property generated by $\mathcal{X}$ inductively by $L_n(\mathcal{X}) = L(L_{n-1}(\mathcal{X}))$. If a group is in $L_n(\mathcal{X})$ then the $n$-th successive normal closure of each element in the group is an $\mathcal{X}$-group. The converse of this statement is not necessarily true.

Let $\mathcal{C}$ denote the class of cyclic groups. Consider the group

$$G = \langle \{a, b\} \mid R \rangle$$

where

$$R = \{ a^4 = [b, a, a] = [a^2, b], b^2, [b, a, a, a], [b, a, a, b], [b, a, b], [b, a]^2, [b, a, a]^2 \}. $$

It is straightforward to check that $y^G$ is cyclic for all $y \in G$. However, $([b, a])^G = ([b, a])^{a^2}$ which is isomorphic to $C_2 \times C_2$. It follows that $a^G$ is not in $L(\mathcal{C})$ and hence $G$ is not an $L_2(\mathcal{C})$ group.

Let $n \geq 1$ be an integer. We generalize Theorem 1 by considering three conditions for a group $G$:

(a) $G$ is a central-by-finite $L_n(\mathfrak{A})$-group, where $\mathfrak{A}$ denotes the class of abelian groups,

(b) $G$ has a finite covering by subnormal abelian subgroups of defect at most $n$,

(c) $G$ is a central-by-finite $(n + 1)$-Engel group.

In Section III, we will show that each condition above is a consequence of the preceding one for all $n$. For $n = 2$ the statements are equivalent since the normal closure of each element of any 3-Engel group is 2-Engel ([5]) and in turn the normal closure of each element in any 2-Engel group is abelian. Hence the class of 3-Engel groups coincide with $L_2(\mathfrak{A})$. Similarly, we will show for center-by-metabelian groups all three conditions are equivalent for any $n$.

A group $G$ is called $n$-abelian if for all $x, y$ in $G$ $(xy)^n = x^ny^n$. A group $G$ is called $n$-central if for all $x, y$ in $G$ $[x^n, y] = 1$. In Sections
IV and V, we will provide characterizations for groups having a finite covering by subnormal 3-abelian subgroups and subnormal 2-central subgroups. In groups covered (not necessarily finitely) by subnormal 3-abelian or 2-central subgroups, certain Engel conditions hold. In the case of metabelian groups, these Engel conditions also imply the group has such a covering (Theorems 6 and 9). These coverings are shown to be finite if and only if these metabelian groups also possess certain finiteness conditions (Corollaries 1 and 2).

We conclude this section with B. H. Neumann’s Lemma which is fundamental to the study of finite coverings.

**Lemma 1 ([11, 4.4])**  Let \( G = \bigcup_{i=1}^{k} H_i g_i \) where the \( H_i \) are (not necessarily distinct) subgroups of \( G \). Then if we omit from the union those cosets \( H_i g_i \) for which \( [G : H_i] \) is infinite, the union of the remaining cosets is still all of \( G \).

### II. Notation and preliminary results

Our notation is standard. If \( G \) is a group and \( h, g \in G \), then \( h^g = g^{-1} h g \), and for any subsets \( X, Y \) of \( G \) we will shall write \( X^Y = \langle x^y \mid x \in X, y \in Y \rangle \). For elements \( x, y, x_i \) in \( G \), \( [x, y] = x^{-1} y^{-1} x y \) is the commutator of \( x \) and \( y \) and

\[
[x_1, \ldots, x_{n+1}] = ([x_1, \ldots, x_n], x_{n+1}].
\]

For \( n = 0 \), \( [x, 0 y] = x \) and for \( n \geq 1 \) we denote the \( n \)-Engel word by \( [x, y, \ldots, y]_n \). The set of right 2-Engel elements is defined as

\[
L(G) = \{ a \in G \mid [a, x, x] = 1 \text{ for all } x \in G \}.
\]

The following lemma is a collection of facts about right 2-Engel elements found in [9]. The lemma as stated can be found in [4, Lemma 5].

**Lemma 2**  Let \( a \in L(G) \). Then:

(i) \( [x, 2, a] = 1 \) for all \( x \in G \);

(ii) \( [x, y, x, a] = 1 \) for all \( x, y \in G \);

(iii) \( [x, a^k] = [x^k, a] = [x, a]^k \) for all \( x \in G, k \in \mathbb{Z} \).
The following commutator identities will be used without further reference:
\[
[ab, c] = [a, c][a, b][b, c] = [a, c][a, c, b][b, c] \\
[a, bc] = [a, c][a, b]c = [a, c][a, b][a, b, c] \\
[a^{-1}, b]a[a, b] = 1, \quad [a, b][b, a] = 1.
\]

If \(X, Y\) are subsets of \(G\), then \([X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle\). We denote the derived subgroup of \(G\) by \(G' = [G, G]\). A group is metabelian if its derived subgroup is abelian i.e. \([G', G'] = 1\).

For subsets \(X_0, X_1, \ldots, X_{n+1}\) of \(G\) we define \([X_0] = \langle X_0 \rangle\) and inductively
\[
[X_0, X_1, \ldots, X_{n+1}] = [[X_0, X_1, \ldots, X_n], X_{n+1}], \quad \text{for } n \geq 1.
\]

We write \([X, Y, \ldots, Y] = [X, n Y]\).

Let \(H\) be a subgroup of the group \(G\). The series of successive normal closures of \(H\) in the group \(G\) is defined as: \(H^{G, 0} = G\) and \(H^{G, i+1} = HH^{G, i}\). The subgroup \(H\) is subnormal of defect \(n\) or \(n\)-subnormal if \(n\) is the smallest integer such that \(H^{G, n} = H\).

The following lemma, which can be found in [10], summarizes some facts about subnormal subgroups.

**Lemma 3**  Let \(G\) be a group and \(n \geq 0\) an integer. Suppose \(H\) is a subgroup of \(G\). Then

(i) \(H\) is \(n\)-subnormal in \(G\) if and only if \([G, n H] \leq H\),
(ii) If \(H\) is \(n\)-subnormal and \(N \triangleleft G\), then \(HN\) is subnormal of defect at most \(n\) in \(G\).

The following result will be used to characterize metabelian groups in \(L(\mathfrak{X})\) where \(\mathfrak{X}\) is a property defined by powers of commutator words. The proof can be found in [7].

**Lemma 4**  Let \(G\) be a metabelian group and \(n, k\) integers with \(n \geq 1\). Then the following statements are equivalent:

(i) \([w_1, w_2, \ldots, w_{n+1}]^k = 1\) for all \(w_i \in x^G\), for all \(x \in G\);
(ii) \([z, v]^k = 1\) for all \(z, v \in x^G\), for all \(x \in G\);
(iii) \([g, n+1 x]^k = 1\) for all \(g, x \in G\).
We conclude this section by showing that center-by-metabelian 
\((n + 1)\)-Engel groups are exactly those groups whose normal closures 
are nilpotent of class at most \(n\).

**Theorem 2**  
Let \(G\) be a center-by-metabelian group. Then the following statements are equivalent:

(i) \(x^G\) is nilpotent of class at most \(n\) for all \(x \in G\);
(ii) \(x^G\) is \(n\)-Engel for all \(x \in G\);
(iii) \(G\) is \((n + 1)\)-Engel.

The following lemma will facilitate the proof of the theorem.

**Lemma 5**  
Let \(G\) be a center-by-metabelian group. Let \(a, a_i \in G\), \(u, v \in G'\), and \(j \geq 1\) an integer. Then the following commutator equalities hold:

(i) \([u, av] = [u, a]z\) for some \(z \in Z(G)\).
(ii) \([uv, a_1, \ldots, a_j] = [u, a_1, \ldots, a_j]^w[v, a_1, \ldots, a_j]\) for some \(w \in G'\).
(iii) \([u^{-1}, a_1, \ldots, a_j] = [u, a_1, \ldots, a_j]^{-w}\) for some \(w \in G'\).

**Proof.** We prove the lemma using the usual commutator identities and the commutator law 
\([x_1, x_2], [x_3, x_4], x_5] = 1\) 
which holds for center-by-metabelian groups.

(i) This follows by the usual commutator expansion modulo \(Z(G)\). 
For \(j = 1\), we have \([uv, a_1] = [u, a_1]^w[v, a_1]\). Setting \(v = w\) shows the 
equality holds. Assume that the equality holds for \(j\). Then 
\([uv, a_1, \ldots, a_j] = [u, a_1, \ldots, a_j][v, a_1, \ldots, a_j]z\) 
for some \(z \in Z(G)\). Thus 
\([uv, a_1, \ldots, a_{j+1}] = [u, a_1, \ldots, a_j][v, a_1, \ldots, a_j]z, a_{j+1}\].
The usual commutator expansion yields 
\([uv, a_1, \ldots, a_{j+1}] = [u, a_1, \ldots, a_{j+1}]^w[v, a_1, \ldots, a_{j+1}]\] 
where \(w = [v, a_1, \ldots, a_j]\).

(ii) This is an immediate consequence of (ii) by observing 
\(1 = [uu^{-1}, a_1, \ldots, a_j] = [u, a_1, \ldots, a_j]^w[u^{-1}, a_1, \ldots, a_j].\)
Proof of Theorem 2. Statement (i) implies (ii), and (ii) implies (iii). Therefore we complete the proof by showing (iii) implies (i). Let $G$ be a center-by-metabelian $(n + 1)$-Engel group. Suppose $a \in G$. We will show that $a^G$ is nilpotent of class at most $n$. By Lemma 5 (iii) we have for any $g \in G$

$$[a^{g^m}, a] = [a, g^m a] = [[g, a]^{-1}, a] = [g, n+1]^{-w} = 1.$$  

Suppose for $k < n$ we have $[a^{g_1}, \ldots, a^{g_k}, a^{g_{n-k+1}}] = 1$. Then by Lemma 5 (i) and the induction hypothesis we have with $z \in Z(G)$,

$$[a^{g_1}, \ldots, a^{g_k}, a^{n+1}] = [a^{g_1}, \ldots, a^{g_k}, a^{n+1}] = 1.$$  

Let $k = n$ and it follows $a^G$ is nilpotent of class at most $n$. □

III. Finite coverings by subnormal abelian subgroups

In this section we will show that condition (a) implies (b), and (b) implies (c) for all groups and that all three conditions are equivalent for center-by-metabelian groups.

Theorem 3 Let $G$ be a group and $n \geq 1$ an integer.

(i) If $G$ is a central-by-finite $L_n(A)$ group then $G$ has a finite covering by subnormal abelian subgroups of defect at most $n$.

(ii) If $G$ has a finite covering by subnormal abelian subgroups of defect at most $n$ then $G$ is a central-by-finite $(n+1)$-Engel group.

Proof. (i) Suppose $G$ is a central-by-finite $L_n(A)$ group. We proceed by induction on $n$, with the cases $n = 1$ and $n = 2$ known to be true by Theorem B and Theorem 1 respectively. Assume the result is true for $n-1$ and $G$ is a central-by-finite $L_n(A)$-group. Then for all $x \in G$, $x^G$ is an $L_{n-1}(A)$-group, in particular for $g_i^G$ where $\{g_1, \ldots, g_k\}$ is a transversal of $Z(G)$ in $G$. Then the subgroups $K_i = g_i^G Z(G)$ are normal in $G$ and $G = \cup_{i=1}^k K_i$. Let $h$ be an element
of $K_i$ and let $u, u' \in \mathfrak{h}_{K_i, n-1}$. Then $u$ and $u'$ have the form $gv$ where $g \in g_i^{G,n}$ and $v \in Z(G)$. By hypothesis all elements in $g_i^{G,n}$ commute. Hence $u$ and $u'$ commute and we have that $K_i$ is an $L_{n-1}(\mathfrak{A})$-group. Furthermore, $Z(G) \subseteq K_i$ implies the $K_i$ is central-by-finite. By the induction hypothesis, $K_i$ has a finite covering by subnormal abelian subgroups of defect at most $n-1$. Therefore $K_i = \bigcup_{j=1}^{n_i} H_{ij}$. Thus $G = \bigcup_{i,j} H_{ij}$ is a finite covering of $G$ by subnormal abelian subgroups of defect at most $n$.

(ii) Suppose $G$ has a finite covering by subnormal abelian subgroups of defect at most $n$. By Baer’s result we have $G$ is central-by-finite. Let $g, x$ be elements in $G$. By hypothesis there exists a subnormal abelian subgroup $H$ of defect at most $n$ such that $x \in H$. By Lemma 3, $[G_n, H] \leq H$. In particular, $[g, n x]$ is an element of $H$ and hence $[g, n+1 x] = 1$ since $H$ is abelian. Therefore $G$ is $(n+1)$-Engel. □

We note that there exist $L_n(\mathfrak{A})$-groups which are not central-by-finite and hence do not have finite coverings by abelian subgroups even for $n = 1$. For example consider the unipotent upper triangular $3 \times 3$ matrices over $\mathbb{Z}$. This group is nilpotent of class 2 and hence in $L(\mathfrak{A})$. However it is not central-by-finite.

All $L_n(\mathfrak{A})$-groups are $(n+1)$-Engel groups by Lemma 3 but the converse is not always true (see [12, Example 4.3]). However, for center-by-metabelian groups the property $(n+1)$-Engel does imply $L_n(\mathfrak{A})$ by repeated applications of Theorem 2 and hence conditions (a), (b), and (c) are equivalent for this class of groups:

**Theorem 4** Let $G$ be a center-by-metabelian group. Then the following statements are equivalent:

(i) $G$ is a central-by-finite $L_n(\mathfrak{A})$-group;

(ii) $G$ has a finite covering by subnormal abelian subgroups of defect at most $n$;

(iii) $G$ is a central-by-finite $(n+1)$-Engel group.
IV. Finite coverings by subnormal 3-abelian subgroups.

In this section we consider groups with finite coverings by subnormal 3-abelian subgroups. Analysis similar to that in Section III again yields results relating such coverings to iterated Levi-properties and Engel conditions.

Let \( n \) be an integer. Two elements \( x, y \) in a group \( G \) \( n \)-commute if \((xy)^n = x^ny^n \) and \((yx)^n = y^nx^n \). A group \( G \) is \( n \)-abelian if any two elements in the group \( n \)-commute. The class of \( n \)-abelian groups is denoted by \( A_n \). The \( n \)-center of a group, denoted by \( Z(G,n) \), is defined to be the set of those elements which \( n \)-commute with every element in the group. This set forms a characteristic subgroup of \( G \) \[8\].

The following three conditions are analogous to \( (a) \), \( (b) \), and \( (c) \) in the introduction.

\((a')\) \([G : Z(G,3)]\) is finite and \( G \) is an \( L_n(\mathfrak{A}_3) \)-group,

\((b')\) \( G \) has a finite covering by subnormal 3-abelian subgroups of defect at most \( n \),

\((c')\) \([G : Z(G,3)]\) is finite, \( G \) is an \((n+2)\)-Engel group, and \( G \) satisfies the identity \([x,y_n+1]^3 = 1\).

We will prove each condition is implied by the preceding one for all groups.

**Theorem 5** Condition \((a')\) implies \((b')\) and \((b')\) implies \((c')\) for all groups.

**Proof.** Assume \([G : Z(G,3)]\) is finite and \( G \) is in \( L_n(\mathfrak{A}_3) \). Then for all \( x \) in \( G \), \( x^{G,n} \) is a subnormal 3-abelian subgroup of defect at most \( n \). It follows from [2, Theorem 3.3] that \( Z(G,3)x^{G,n} \) is 3-abelian, and that \( Z(G,3)x^{G,n} \) is subnormal of defect at most \( n \) in \( G \). Let \( \{x_1, \ldots, x_k\} \) be a transversal of \( Z(G,3) \) in \( G \). Then \( G = \bigcup_{i=1}^k Z(G,3)x_i^{G,n} \) is the desired finite covering of \( G \).

Assume that \( G = \bigcup_{i=1}^k H_i \) has a finite covering by subnormal 3-abelian subgroups of defect at most \( n \). Then by [2, Theorem 3.2(2)] we have \([G : Z(G,3)]\) is finite. Let \( y \) be any element of \( G \). Then \( y^{G,n} \)
being 3-abelian is equivalent to the group being 2-Engel and the derived subgroup having exponent 3 (see [6, Theorem B]). Hence for all \( h \in y^{G,n} \), \( h^{y^{G,n}} \) is abelian. In particular, \( y^{G,n+1} \) is abelian. By Lemma 3 we have \( [G_{m+1} y^{G,n+1}] \leq y^{G,n+1} \). It follows that \( G \) is \((n + 2)\)-Engel. We again use Lemma 3 to see that \([G_{m+1} y^{G,n}]\) is contained in the derived subgroup of \( y^{G,n} \). Hence \([x_{m+1} y]^{3} = 1\) as needed. □

The following theorem is analogous to Theorem 2 in that the iterated Levi-property generated by 3-abelian for metabelian groups can be characterized by an Engel condition and certain commutator laws.

**Theorem 6** Let \( G \) be a metabelian group and \( n \geq 1 \) an integer. Then the following statements are equivalent:

(i) \( G \in L_{n}(\mathfrak{A}_{3}) \);
(ii) \( G \) satisfies the law \([x_{m-1} y]^{3}, y] = [x_{m-1} y, y^{3}] = [x_{m} y]^{3} \);
(iii) \( G \) is \((n + 2)\)-Engel and \([x_{m+1} y]^{3} = 1\) for all \( x, y \in G \).

**Proof.** We prove (i) implies (ii) and (iii) implies (i) by induction on \( n \). For \( n = 1 \) the theorem is true for all groups by [6, Theorem 1]. Suppose the result is true for \( n - 1 \).

(i) implies (ii): Let \( G \in L_{n}(\mathfrak{A}_{3}) \). Then the normal closure of each element in \( G \) is in \( L_{n-1}(\mathfrak{A}_{3}) \). By the induction hypothesis, for an arbitrary \( y \in G \) the identities \([u_{m-2} v]^{3}, v] = [u_{m-2} v, v^{3}] = [u_{m-1} v]^{3}\) hold for \( u, v \in y^{G} \). Hence in particular for \( u = [x, y] \) and \( v = y \) we have (ii).

(iii) implies (i): Suppose \( G \) is \((n + 2)\)-Engel and \([x_{m+1} y]^{3} = 1\) for all \( x, y \in G \). Using Lemma 4 we have for each element \( x \) in \( G \), \( x^{G} \) is \((n + 1)\)-Engel and \([u_{n} v]^{3} = 1\) for all \( u, v \in x^{G} \). By the induction hypothesis, \( x^{G} \) is in \( L_{n-1}(\mathfrak{A}_{3}) \). Hence \( G \in L_{n}(\mathfrak{A}_{3}) \) and \( G \) satisfies (i).

We complete the theorem by showing (ii) implies (iii). Suppose \( G \) satisfies the law \([x_{m-1} y]^{3}, y] = [x_{m-1} y, y^{3}] = [x_{m} y]^{3}\). Using the fact that \( G \) is metabelian we get

\([x_{m} y]^{3} = [x_{m-1} y, y^{3}] = [x_{m} y]^{3}[x_{m+1} y]^{3}[x_{m+2} y] \]

and hence \( 1 = [x_{m+1} y]^{3}[x_{m+2} y] \). Substituting \( y^{-1} \) for \( y \) in the iden-
ty above we get
\[ 1 = [x_{n+1} y^{-1}]^3 [x_{n+2} y^{-1}] = [x_{n+1} y]^{3(-1)^{n+1} y^{-(n+1)}} [x_{n+2} y]^{(-1)^{n+2} y^{-(n+2)}}. \]

Conjugating with \( y^{n+2} \) yields
\[ 1 = [x_{n+1} y]^{-3} [x_{n+2} y] = [x_{n+1} y]^{-3} [x_{n+2} y]^{-3} [x_{n+2} y]. \]

Thus we obtain \( 1 = [x_{n+1} y]^{-3} [x_{n+2} y]^{-2} \) which implies \( 1 = [x_{n+2} y] \) and hence \( 1 = [x_{n+2} y]^{-3} \) which establishes (iii). \( \square \)

The following corollary is an immediate consequence of Theorem 5 and Theorem 6.

**Corollary 1** Let \( G \) be a metabelian group. The statements (a'), (b'), and (c') are equivalent.

### V. Finite coverings by subnormal 2-central subgroups

A group \( G \) is \( n \)-central if it satisfies the identity \([x^n, y] = 1\) for all \( x, y \in G \). We denote the class of \( n \)-central groups by \( \mathfrak{Z}_n \). The following three conditions are analogous to (a), (b), and (c) above.

- \((a'')\) \([G : T]\) is finite, where \( T = L(G) \cap C_G(G^2) \) and \( G \) is an \( L_n(\mathfrak{Z}_2) \) group.
- \((b'')\) \( G \) has a finite covering by subnormal 2-central subgroups of defect at most \( n \),
- \((c'')\) \([G : T]\) is finite, \( G \) is an \((n+2)\)-Engel group, and \( G \) satisfies the identity \([x_{n+1} y]^2 = 1\).

We first show each condition is implied by the preceding one for all groups.

**Theorem 7** Condition \((a'')\) implies \((b'')\) and \((b'')\) implies \((c'')\).

**Proof.** Suppose \([G : T]\) is finite and \( G \in L_n(\mathfrak{Z}_2) \). Then for all \( x \) in \( G \), \( x^{G,n} \) is a subnormal 2-central subgroup of defect at most \( n \). By Lemma 3, \( T x^{G,n} \) is subnormal of defect at most \( n \) as well. We will show \( T x^{G,n} \) is 2-central.
Consider the subgroup $L(G)\times x^{G,n}$. We claim $L(G)\times x^{G,n}$ is 2-Engel. To see this, we let $a, b \in L(G)$ and $u, v \in x^{G,n}$. Then

$$[ua, 2vb] = [u, 2vb]$$
$$= [u, b][u, v]^b, vb]$$
$$= [(u, b), vb]^b[(u, v)^b, vb]$$
$$= [u, b, v]^b[u, v]^b, b][[(u, v)^b, v]^b$$
$$= [u, b, v]^b[u, v, b^{-1}]b^2[u, v, v]^b[u, v, b]$$
$$= [u, b, v]^b[u, v, b][u, v, b^{-1}]b^2[u, v, v]^b.$$ 

Thus it suffices to show that if $b \in L(G)$ and $u, v \in x^{G,n}$, then $[u, b, v] = [u, v, b] = 1$. Let $u = \prod_{i=1}^{n} x^{g_i}$ and $v = \prod_{j=1}^{m} x^{h_j}$ where $\delta_i, \epsilon_j = \pm 1$ and $g_i, h_j \in x^{G,n-1}$. Expanding $[u, b, v]$ shows that it is a product of conjugates of commutators of the form $[x^{g_i}, b, x^{h_j}]$ where $g_i, h_j \in x^{G,n-1}$. Without loss of generality we may show that $[x^{g_i}, b, x] = 1$. But this follows immediately from Lemma 2. Similarly, for $[u, v, b]$, it suffices to show that $[x^{g_i}, x, b] = 1$. This identity follows by expanding $[b, [x^{g_i}, x]]$, its inverse, and using Lemma 2. Thus $L(G)\times x^{G,n}$ is 2-Engel, and since $T \subseteq L(G)$, $Tx^{G,n}$ is 2-Engel as well.

Now let $a, b \in T$ and $u, v \in x^{G,n}$. Then

$$[au, (bv)^2] = [u, (bv)^2] = [u^2, bv] = [u^2, v] = 1.$$ 

Thus $Tx^{G,n}$ is 2-central for all $x \in G$. Choose a transversal of $T$ in $G$ and $b''$ follows.

Assume that $G = \bigcup_{i=1}^{k} H_i$ with $H_i$ subnormal of defect at most $n$ and $H_i$ in $3_2$. By [2], we conclude $[G : T]$ is finite. The 2-central groups are exactly those 2-Engel groups with derived subgroup of exponent 2 [7, Theorem 7]. For any $y$ in $G$, it is contained in some $H_i$, and hence $y^{G,n} \leq H_i$ and it $y^{G,n+1}$ is abelian. By Lemma 3 $[G_{n+1} y^{G,n+1}] \leq y^{G,n+1}$ and hence $G$ is $(n + 2)$-Engel. Similarly, $[G_{n+1} y^{G,n}]$ is in the derived subgroup of $y^{G,n}$. If follows that $[x_{n+1} y^{G,n}] = 1$. □

We next show that for $n = 1$ all three conditions above are equivalent.
Theorem 8 A group $G$ is an $L(Z_2)$ group with $[G : T]$ is finite, where $T = L(G) \cap C_G(G^2)$ if and only if $[G : T]$ is finite, $G$ is 3-Engel, and $G$ satisfies the law $[x, y, y]^2$.

Proof. By [7, Theorem 8], a group $G$ is 3-Engel and satisfies the law $[x, y, y]^2$ if and only if the normal closure of each element of $G$ is 2-central. Hence, $G$ is in $L(Z_2)$ as needed. □

We now characterize the iterated Levi-property generated by 2-central in metabelian groups by an Engel condition and a commutator law.

Theorem 9 Let $G$ be a metabelian group and $n \geq 1$ an integer. Then the following statements are equivalent:

(i) $G \in L_n(Z_2)$;
(ii) $G$ satisfies the identity $[[x, n y]^2, y] = [x, n y, y^2] = [x, n+1 y]^2 = 1$;
(iii) $G$ is $(n + 2)$-Engel and $[x, n+1 y]^2 = 1$ for all $x, y \in G$.

Proof. We prove the (i) implies (ii) and (iii) implies (i) by induction on $n$. For $n = 1$ the result is true for all groups by [7, Theorem 8]. Suppose the result is true for $n - 1$.

(i) implies (ii): Suppose $G \in L_n(Z_2)$. Then the normal closure of each element in $G$ is in $L_{n-1}(Z_2)$. By the induction hypothesis, for an arbitrary $y \in G$ the identities $[[u, n-1 v]^2, v] = [u, n-2 v, v^2] = [u, n-1 v]^2$, and $[u, n v]^2 = [u, n+1 v] = 1$ hold for $u, v \in y^G$. In particular, for $u = [x, y]$ and $v = y$ we see (ii) and (iii) hold.

(iii) implies (i): Conversely, suppose $G$ is $(n + 2)$-Engel and $[x, n+1 y]^2 = 1$ for all $x, y \in G$. Using Lemma 4 we have for each element $x$ in $G$, $x^G$ is $(n + 1)$-Engel and $[u, n v]^3 = 1$ for all $u, v \in x^G$. By the induction hypothesis, $x^G$ is in $L_{n-1}(Z_2)$. Hence $G \in L_n(Z_2)$ and $G$ satisfies (i).

We complete the theorem by showing (ii) implies (iii). Suppose $G$ satisfies the identity $[[x, n y]^2, y] = [x, n y, y^2] = [x, n+1 y]^2 = 1$. Then expanding the commutator $[x, n y, y^2]$ we get $[x, n y, y^2] = [x, n+1 y]^2 [x, n+2 y]$. It follows from $[x, n+1 y]^2 = 1$ then $G$ is $(n + 2)$-Engel as needed. □
The following corollary is an immediate consequence of Theorem 7 and Theorem 9.

**Corollary 2** Let $G$ be a metabelian group. Then statements $(a'')$, $(b'')$, and $(c'')$ are equivalent.

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**References**