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Groups with 3-abelian normal closures

By

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Dedicated to F. W. Levi on the hundredth anniversary of his birth

1. Introduction and Results. The group theoretical properties called Levi-properties which were introduced in [6] are modeled after the groups first classified by F. W. Levi [10] where conjugates commute. Given a property $\mathfrak{E}$, the derived property $L(\mathfrak{E})$ will be called a Levi-property generated by $\mathfrak{E}$, if and only if the normal closures $\langle x^G \rangle$ of all elements x in a group G have property $\mathfrak{E}$. For $\mathfrak{E}$ = abelian the derived property $L(\mathfrak{E})$ is the class of 2-Engel groups investigated by Levi. It is less obvious that the 3-Engel condition is also a Levi-property.

Theorem A. ([8]). *The following conditions for a group G are equivalent*:

(i) *G is a 3-Engel group.*
(ii) *The normal closures $\langle x^G \rangle$ of all elements x in G are 2-Engel groups.*
(iii) *The normal closures $\langle x^G \rangle$ of all elements x in G are nilpotent of class at most 2.*

In [1], R. Baer calls a group n-abelian if it satisfies the identity $(xy)^n = x^n y^n$ for an integer n. In the same paper Baer suggests the following generalization principle: "From any concept and property involving the fact that certain elements (or functions of elements) commute, one may derive new concepts and properties by substituting everywhere n-commutativity." Following Baer's idea in the case of the Levi-property $L(\mathfrak{E})$ for $\mathfrak{E}$ = abelian leads to the consideration of groups with n-abelian normal closures.

If G is a group with n-abelian normal closures it follows easily that we have for all $x, y \in G$

(1.1) $[x, y^n] = [x, y]^n$ for the integer n.

An immediate consequence of (1.1) is the identity

(1.2) $[x, y^n] = [x^n, y]$ for the integer n.

Groups satisfying (1.1) or (1.2) were introduced in [7], and they were called n Levi groups or n-Bell groups respectively. There are structural similarities between n-abelian groups and n-Bell groups. These similarities were the topic of investigation in [2].

It is well-known that for small values of n the conditions that a group has exponent n or is n-abelian put restrictions on the nilpotency class of a group. Of particular interest in this context is the following result on 3-abelian groups, due to Levi [11].

Theorem B. *A group G is 3-abelian if and only if G is a 2-Engel group and* $(G_2)^3 = 1$.

This trend of nilpotency restrictions for small values of n is continued when it comes to n-Bell groups. For 3-Bell groups the following result is obtained in [2; Theorem $E(a)$].

Theorem C. *A 3-Bell group is nilpotent of class at most 6 and satisfies the identity* $[x, y, y, z, z] = 1$, *hence is a 4-Engel group.*

An immediate question is whether the nilpotency class for 3-Bell groups given in Theorem C is sharp. As a consequence of the main result of this paper it will be shown that the exact bound for the nilpotency class of 3-Bell groups is 4.

Since any group with n-abelian normal closures is an n-Bell group, any result on the latter class of groups sheds light on the structure of groups with the Levi-property generated by n-abelian. However, an n-Bell group is not necessarily n-Levi, as pointed out in examples in [7] and [2]. Thus, in general, the conditions n-abelian normal closures, n-Levi, and n-Bell are not equivalent for a group G and an arbitrary value of n. For $n = 2$ it can be easily seen that these three conditions are equivalent, since each of them is equivalent to the condition 2-Engel. The main result of this paper is that these conditions are equivalent for any group G if $n = 3$. We have the following theorem:

Theorem 1. *For any group G the following four conditions are equivalent:*

(i) *The normal closures* $\langle x^G \rangle$ *of all elements x in G are 3-abelian.*
(ii) *G is a 3-Levi group.*
(iii) *G is a 3-Bell group.*
(iv) *G is a 3-Engel group satisfying the identity* $[x, y, y]^3 = 1$ *for all* $x, y \in G$.

As in Theorem B, the case of 3-abelian groups, we obtain restrictions for the nilpotency class and for exponents of the terms of the lower central series of groups considered in Theorem 1.

Theorem 2.

(A) *Let G be a 3-Bell group. Then* $(G_3)^9 = (E_2(G))^3 = (G_4)^3 = G_5 = 1$.
(B) *There exists a 3-Bell group H with* $H_4 \neq 1$ *and* $(H_3)^3 \neq 1$.

Theorem 1 and 2 suggest a possible connection between groups with 3-abelian normal closures and the groups satisfying the identity $[y, x, x, z] = 1$, a class of 3-Engel groups investigated by Heineken in [4]. However, as can be shown by examples, this is not the case. A group satisfying the identity $[y, x, x, z] = 1$ does not necessarily have 3-abelian normal closures. The converse is not true either.

N o t a t i o n. The notation is for the most part standard. We give a partial list for the convenience of the reader.

$\langle X \rangle$ = subgroup generated by the set X,
$x^g = g^{-1} x g$,

$\langle x^G \rangle = \langle x^g \mid g \in G \rangle$ the normal closure of x in G,
$[y, x] = y^{-1} x^{-1} y x$ and $[x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$,
$[y,_n x] = [[y,_{n-1} x], x]$ and $[y,_1 x] = [y, x]$,
$[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle$,
$G' = [G, G]$ and $G'' = [G', G]$,
$G_n = [G_{n-1}, G]$ and $G_1 = G$,
$E_2(G) = \langle [x, y, y] \mid x, y \in G \rangle$.

The following commutator identities will be used frequently without special reference:

$$[ab, c] = [a, c]\,[a, c, b]\,[b, c]$$

$$[a, bc] = [a, c]\,[a, b]\,[a, b, c]$$

$$1 = [a, b, c^a]\,[c, a, b^c]\,[b, c, a^b].$$

2. Some preparatory lemmas. The first lemma we state without proof.

Lemma 1.

(i) *If G is 3-abelian then $[x^3, y] = [x, y]^3 = 1$ for all $x, y \in G$.*
(ii) *If G is 3-Bell then $[x^3, y, y] = [x, y^3, y] = 1$ for all $x, y \in G$.*

The proof of the first part can already be found in [11]. For part (ii), see Lemma 1(b) in [2].

The following two lemmas facilitate the proof of Theorem 1 by placing some of the necessary commutator calculations ahead of the main theorem.

Lemma 2. *Let G be a 4-Engel group of nilpotency class 5 satisfying the identity $[x, y^3, y] = 1$. Then $[x, y, y]^3 = 1$ for all $x, y \in G$.*

Proof. Using the nilpotency restriction on G we obtain the following expansion:

$$[x, y^3] = [x, y]^3\,[x, y, y]^3\,[x, y, y, [x, y]]\,[x, y, y, y]. \tag{2.1}$$

Commuting (2.1) with y and observing that G is a 4-Engel group of class 5 yields

$$1 = [x, y, y]^3\,[x, y, y, y]^3\,[x, y, y, [x, y]]^3. \tag{2.2}$$

Commuting (2.2) with $[x, y]$ gives $1 = [x, y, y, [x, y]]^3$, hence

$$1 = [x, y, y]^3\,[x, y, y, y]^3. \tag{2.3}$$

By commuting (2.3) with y we get

$$1 = [x, y, y, y]^3.$$

Finally, this together with (2.3) yields

$$1 = [x, y, y]^3.$$

Lemma 3. *Let G be a 3-Bell group of class 5. Then G is a 3-Engel group.*

Proof. Since G is 3-Bell, it folllows from Theorem C that G satisfies the identity

$$1 = [y, x, x, z, z]. \tag{2.4}$$

By Lemma 1 (ii) we have $[x, y^3, y] = 1$ for all $x, y \in G$. Thus by applying Lemma 2 we get

$$1 = [x, y, y]^3. \tag{2.5}$$

Next we expand $1 = [x^3, y][y^3, x]$ with the help of (2.1). By using (2.4), (2.5) and the class restriction on G we thus obtain

$$1 = [x, y, y, y][y, x, x, x][x, y, y, [x, y]][y, x, x, [y, x]]. \tag{2.6}$$

Substitution of x^2 for x into (2.6) leads to

$$1 = [x^2, y, y, y][y, x^2, x^2, x^2][x^2, y, y, [x^2, y]][y, x^2, x^2, [y, x^2]].$$

Because of the class restriction on G and (2.4) we can expand this identity linearly and get

$$1 = [x, y, y, y]^2 [y, x, x, x]^2 [x, y, y, [x, y]]^4 [y, x, x, [y, x]]^8.$$

This together with (2.5) yields

$$1 = [x, y, y, y]^{-1} [y, x, x, x]^{-1} [x, y, y, [x, y]] [y, x, x, [y, x]]^{-1}.$$

By combining the above identity with (2.6) and observing (2.5), we obtain

$$1 = [x, y, y, [x, y]].$$

This together with (2.6) implies

$$1 = [x, y, y, y][y, x, x, x]. \tag{2.7}$$

Commuting (2.7) with y together with (2.4) lead to

$$1 = [y, x, x, x, y]. \tag{2.8}$$

Substitution of yx for x into (2.7) gives

$$1 = [x, y, y, y][y, x, yx, yx].$$

Further expansion of this identity and observing (2.4), (2.7), (2.8), and $G_6 = 1$ lead to

$$1 = [y, x, x, y][y, x, y, x][y, x, y, y][y, x, x, y, x][y, x, y, x, y]. \tag{2.9}$$

Commuting (2.9) with y together with (2.4) give

$$1 = [y, x, y, x, y]. \tag{2.10}$$

Thus (2.9) becomes

$$1 = [y, x, x, y][y, x, y, x][y, x, y, y]. \tag{2.11}$$

By substitution of x^2 for x into (2.11) we obtain

$$1 = [y, x, x, y]^4 [y, x, y, x]^4 [y, x, y, y]^2.$$

This together with (2.5) and the class restriction leads to

$$1 = [y, x, x, y]\,[y, x, y, x]\,[y, x, y, y]^2.$$

Combining this identity with the inverse of (2.11) finally yields $1 = [x, y, y, y]$. Thus G is a 3-Engel group. This concludes the proof of Lemma 3.

The following proposition allows us to make a reduction to the case of finite groups in the proofs of Theorems 1 and 2.

Proposition 1. *Let $\mathfrak{V}$ be a variety of groups and let $\mathfrak{U}$ be a $\{Q, S\}$-closed class of groups. Assume that every finitely generated $\mathfrak{U}$-group is residually finite. If $\mathfrak{V}$ contains every finite $\mathfrak{U}$-group, then $\mathfrak{U} \subseteqq \mathfrak{V}$.*

3. Proof of the Theorems. An elementary proof of Theorem B using commutator calculations can be found in [11]. The proof presented here follows along the same lines as the proof of Theorem 1. We need the following expansion formula for powers which can be found in [5]:

Let G be a metabelian group. Then

$$(3.1) \qquad (v w^{-1})^n = v^n \prod_{0<i+j<n} [v,_i w,_j v]^{\binom{n}{i+j+1}} w^{-n}$$

for all $v, w \in G$ and all integers $n > 0$.

Proof of Theorem B. Suppose G is 2-Engel and $(G_2)^3 = 1$. Since G is metabelian we can use (3.1) for $n = 3$ and obtain for $v = a$, $w = b^{-1} \in G$

$$(ab)^3 = a^3 [a, b^{-1}]^3 [a, b^{-1}, a]\,[a, b^{-1}, b^{-1}]\, b^3.$$

Since $(G_2)^3 = 1$ and G is 2-Engel if follows immediately that $(ab)^3 = a^3 b^3$ for all $a, b \in G$, thus G is 3-abelian.

Now assume that G is 3-abelian. Then by Lemma 1 (i) $G/Z(G)$ has exponent 3, thus is 2-Engel, and hence $[y, x, x, z] = 1$ for all $x, y, z \in G$. By a result of Heineken in [4] this implies $(G_4)^3 = G_5 = 1$.

Let $a, b \in G$ and consider $H = \langle a, b \rangle \subseteqq G$. The subgroup H is a 2-generator 3-Engel group. Since H has no elements of order 2 it follows by another result of Heineken [3] that $H_4 = 1$. Hence H is of class 3 and therefore metabelian. Thus we may apply (3.1) and obtain for $n = 3$

$$(3.2) \qquad (a b^{-1})^3 = a^3 [a, b]^3 [a, b, a]\,[a, b, b]\, b^{-3}.$$

Since H is 3-abelian we have $(ab^{-1})^3 = a^3 b^{-3}$ and by Lemma 1 (i), we have $[a, b]^3 = 1$ and $b^{-3} \in Z(H)$. Thus (3.2) becomes

$$(3.3) \qquad 1 = [a, b, a]\,[a, b, b].$$

Substituting ab for b into (3.3) and expanding gives $1 = [a, b, a]\,[a, b, b]\,[a, b, a]^b$. This together with (3.3) yields $1 = [a, b, a]^b$ for all $a, b \in G$. Therefore G is 2-Engel. Thus G is metabelian. This together with Lemma 1 (i) implies $(G_2)^3 = 1$. This concludes the proof of Theorem B.

Proof of Theorem 1. Assume G has n-abelian normal closures. Then for any $x, y \in G$ we have $[x^n, y] = x^{-n}(x^y)^n = (x^{-1}x^y)^n = [x, y]^n$. Similarly,

$$[x, y^n] = (y^{-n})^x y^n = [x, y]^n, \quad \text{hence} \quad [x, y^n] = [x^n, y].$$

Thus in particular, if $n = 3$, we have (i) implies (ii), and (ii) implies (iii).

Now assume (iii). By Theorem C, any 3-Bell group is nilpotent, hence any finitely generated 3-Bell group is residually finite. Thus we may apply Proposition 1 with $\mathfrak{U}$ the class of groups satisfying the identity $[x^3, y] = [x, y^3]$. In order to establish (iv) it suffices to show that every finite group in $\mathfrak{U}$ is in the variety $\mathfrak{V}$ defined by the laws $[x, y, y, y] = 1$ and $[x, y, y]^3 = 1$.

Let G be a finite 3-Bell group. Assume first that its order is prime to 3. Since G satisfies the identity $[x^3, y, y] = 1$ of Lemma 1 (ii), it follows that G is nilpotent of class 2, hence trivially in $\mathfrak{V}$.

Next assume G has 3-power order. Theorem C implies that G is nilpotent of class at most 6. Hence $K = G/Z(G)$ is a 3-Bell group of class at most 5. Thus we can use Lemma 3 and conclude that K is a 3-Engel group. As K is a finite 3-group we can apply a result of Heineken [3], and obtain that $K = G/Z(G)$ is nilpotent of class at most 4. Thus G itself is a 3-Bell group of class at most 5. Another application of Lemma 2 and Lemma 3 shows that G is a 3-Engel group satisfying $[x, y, y]^3 = 1$. Hence G is in $\mathfrak{V}$. Thus we have to conclude that (iii) implies (iv).

Using the same argument as above and letting $\mathfrak{V}$ be the variety of groups with $G_5 = 1$, it follows that if G is a 3-Bell group then G is in $\mathfrak{V}$. This result will be stated in Theorem 2 below.

Finally, assume (iv). Since G is a 3-Engel group it follows from Theorem A that all normal closures $\langle y^G \rangle$ of G are nilpotent of class 2, hence 2-Engel. It remains to be shown that $(\langle y^G \rangle')^3 = 1$, then the conclusion follows by Theorem B.

We observe that

$$\langle y^G \rangle' = \langle [u, v] \mid u, v \in \langle y^G \rangle \rangle = \langle [y^g, y]^h; g, h \in G \rangle$$

since $\langle y^G \rangle_3 = 1$. Now $[y^g, y]^{-h} = [g, y, y]^{-h}$ has order 3 by our assumption. It follows from $\langle y^G \rangle'$ being abelian and its generators all having order 3 that $(\langle y^G \rangle')^3 = 1$. By Theorem B it follows that all normal closures of G are 3-abelian. Hence we have (i). This concludes the proof of Theorem 1.

Proof of Theorem 2. (A) As already noted in the proof of Theorem 1, we have $G_5 = 1$ for any 3-Bell group. We observe that G_3 is abelian for any group of class at most 5. Thus it suffices to show $1 = [x, y, u, v]^3 = [x, y, y]^3 = [x, u, v]^9$ for all $x, y, u, v \in G$. Again we can assume without loss of generality that G is a finite 3-group. It follows by Lemma 1 (ii) and [9] that

$$G^3 \subseteqq L(G) \subseteqq Z_3(G),$$

where $L(G) = \{a \in G \mid [a, x, x] = 1 \text{ for all } x \in G\}$ is the characteristic subgroup of 2-right Engel elements of G. This together with $G_5 = 1$ implies $[x, y, u, v]^3 = 1$. That $[x, y, y]^3 = 1$ follows immediately from Lemma 2.

To see that $[x, u, v]^9 = 1$ we expand $1 = [x, uv, uv]^3$ and obtain

(3.4) $$1 = [x, v, u]^3 [x, u, v]^3.$$

Interchanging u and x in (3.4) gives

(3.5) $$1 = [x, u, v]^{-3} [u, v, x]^3.$$

The Jacobi identity together with $(G_4)^3 = 1$ imply

(3.6) $$1 = [x, v, u]^{-3} [x, u, v]^3 [u, v, x]^3.$$

By solving (3.4), (3.5), (3.6) for $[u, v, x]$ we obtain $1 = [u, v, x]^9$, the desired result.

(B) Consider $H = L \times K$ where L is the p-group for $p = 3$ in Example 1 of [6], and K is the group of Example 1 in [2]. The group L is a 3-Engel group of class precisely 4 and $\exp L = 3$. The group K is a 3-group of class 3 with $\exp K_3 = 9$ and $\exp E_2(K) = 3$. Thus G is a 3-Engel group satisfying $[y, x, x]^3 = 1$. Hence, by Theorem 1, all normal closures of H are 3-abelian. However $H_4 = L_4 \neq 1$ and $(H_3)^3 = (K_3)^3 \neq 1$.

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