

On the nonabelian tensor square of a 2-Engel group

By

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1. Introduction. The nonabelian tensor square $G \otimes G$ of a group G is generated by the symbols $g \otimes h$, $g, h \in G$, subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'),$$

for all $g, g', h, h' \in G$, where ${}^g g' = gg'g^{-1}$. Following work of C. Miller [10], R.K. Dennis in [7] introduced the nonabelian tensor square in a search for “homology functors” which are intimately related to K -theory. Subsequently, R. Brown and J.-L. Loday discovered a topological significance for this tensor square [5], [6].

Following the publications of Brown and Loday’s work, a number of purely group theoretic papers have appeared on the topic. Some investigate general structural properties of the tensor square, while others are devoted to explicit descriptions for particular groups, e.g. in [1], [2], [3], [4], [8], and [9]. In [6] it was shown that $G \otimes G$ is finite for G finite, and thus computer methods can be used for the computation of the tensor square. In [4], $G \otimes G$ was computed for all groups G with $|G| \leq 30$, by simplifying the presentation resulting from the definition using a combination of reduction steps. This method is limited in the size of the group for which it can compute the tensor square because of the large number of generators and relations given by the definitions.

In [8], Ellis and Leonard devise a computer algorithm for the computation of tensor products of finite groups which in its applications can handle much larger groups than those given in [4]. Using a CAYLEY-program, they compute the tensor square of $B(2, 4)$, the 2-generator Burnside group of exponent 4, where $|B(2, 4)| = 2^{12}$. Its tensor square is a nilpotent group of class 2 having order 2^{22} .

In this paper we determine the nonabelian tensor square of \mathcal{E} , the free 2-Engel group of rank 3. A group G is a 2-Engel group if $[[x, y], y]$ is a law in G . Our approach is a combination of conventional mathematical derivations and computer aided symbolic computations which implement these derivations for arbitrary elements of \mathcal{E} and $\mathcal{E} \otimes \mathcal{E}$.

The method used here is based on the concept of a crossed pairing which first appeared in [6]. We define it here in the case relevant for tensor squares.

Definition 1.1. *Let G and L be groups. A function $\Phi : G \times G \rightarrow L$ is called a crossed pairing if*

$$(1.1.1) \quad \Phi(gg', h) = \Phi({}^g g', {}^g h)\Phi(g, h),$$

(1.1.2) $\Phi(g, hh') = \Phi(g, h)\Phi({}^h g, {}^h h')$,
for all $g, g', h, h' \in G$.

Crossed pairings allow us to determine homomorphic images of $G \otimes G$ as follows.

Proposition 1.2. ([6]) *A crossed pairing Φ determines a unique homomorphism of groups $\Phi^* : G \otimes G \rightarrow L$ such that $\Phi^*(g \otimes h) = \Phi(g, h)$ for all $g, h \in G$.*

The method involves conjecturing an L and finding a crossed pairing Φ such that Φ^* is an isomorphism. In [2] and [3] this method has been used to determine the nonabelian tensor square of finite and infinite groups, when $G \otimes G$ turns out to be abelian such as when G is nilpotent of class 2 [3; Proposition 3.1]. In this case, the calculations involved are manageable. On the other hand, when $G \otimes G$ fails to be commutative, conjecturing L can be difficult and calculations can quickly become unmanageable by hand.

Since a 2-Engel group is metabelian, it follows from general considerations in [4] that its tensor square is nilpotent at most of class 2 (see Proposition 2.2 below). Using the computer program in [8], Ellis computed for us the tensor square of $B(3, 3)$, the Burnside group of rank 3 and exponent 3, which is a homomorphic image of \mathcal{E} . The tensor square of $B(3, 3)$ is a nonabelian group of order 3^{20} . Hence $\mathcal{E} \otimes \mathcal{E}$ is nonabelian. Analyzing the structure of $B(3, 3) \otimes B(3, 3)$ together with other information we already possessed on the tensor square of 2-Engel groups allowed us to make the right conjecture for L (Example 3.2).

In order to arrive at a mapping Φ which is a crossed pairing, an expansion formula was calculated expressing each $g \otimes h$, $g, h \in \mathcal{E}$, as a product of a finite number of generators of $\mathcal{E} \otimes \mathcal{E}$ (Proposition 3.3). On the basis of this formula a map $\Phi : \mathcal{E} \otimes \mathcal{E} \rightarrow L$ was constructed and verified to be a crossed pairing (Theorem 3.6). As a corollary we obtain $B(3, 3) \otimes B(3, 3)$ as a suitable quotient of $\mathcal{E} \otimes \mathcal{E}$ (Corollary 3.7). In principle, calculations for Proposition 3.3 and Theorem 3.6 can be done manually. However, these calculations involve producing and comparing polynomials with well over 1000 terms. Therefore, a computer program was developed specifically tailored to symbolically expand and collect arbitrary elements in \mathcal{E} and $\mathcal{E} \otimes \mathcal{E}$. This program¹ was written in the GAP programming language [12] making extensive use of GAP's built-in list functions.

In conclusion we want to mention that our method is particularly suitable to determining the tensor square of a torsion free nilpotent group. In a future publication we are planning to compute the nonabelian tensor square of a free group of class 3 of rank 2 and 3 using this method.

¹Information about this program can be obtained by writing to the third author.

2. Some preparatory results. As is customary in context with nonabelian tensor products, we use action on the left, i.e. $[g, h] = ghg^{-1}h^{-1}$ and ${}^g h = ghg^{-1} = [g, h]h$ for group elements g and h . An iterated commutator of weight n is defined recursively as $[g_1, \dots, g_{n-1}, g_n] = [[g_1, \dots, g_{n-1}], g_n]$. For the convenience of the reader, the first lemma lists familiar commutator formulas in the notation used here.

Lemma 2.1. *Let G be a group and $g, g', h, h' \in G$. Then:*

$$(2.1.1) \quad [gg', h] = {}^g [g', h][g, h] = [g, [g', h]][g', h][g, h];$$

$$(2.1.2) \quad [g, hh'] = [g, h] \cdot {}^h [g, h'] = [g, h][h, [g, h']][g, h'];$$

$$(2.1.3) \quad [g, h] = [h, g]^{-1}.$$

The following result in essence already appears in [4], with the additional assumption that G itself is nilpotent.

Proposition 2.2. *Let G be a group. If the derived subgroup G' is nilpotent of class $cl(G')$, then $G \otimes G$ is nilpotent with $cl(G \otimes G) = cl(G')$ or $cl(G') + 1$.*

Proof. If G is abelian, our theorem holds trivially. Thus we can assume that G is nonabelian. By Proposition 2 in [4] there exists a homomorphism κ of $G \otimes G$ onto G' defined by $\kappa(g \otimes h) = [g, h]$, and an action of G on $G \otimes G$ defined by ${}^{g'}(g \otimes h) = {}^{g'}g \otimes {}^{g'}h$, such that the crossed module rules hold for κ .

We prove now by induction on n that for an integer $n > 1$ and $t_1, \dots, t_n \in G \otimes G$

$$(2.2.1) \quad [t_1, \dots, t_n] = [\kappa(t_1), \dots, \kappa(t_{n-1})] \otimes \kappa(t_n).$$

For $n = 2$ this is $[t_1, t_2] = \kappa(t_1) \otimes \kappa(t_2)$, which is simply (11) of Proposition 3 in [4]. Now assume (2.2.1) holds. Then for $t_1, \dots, t_{n+1} \in G \otimes G$ we have

$$\begin{aligned} [t_1, \dots, t_{n+1}] &= [[t_1, \dots, t_n], t_{n+1}] \\ &= [[\kappa(t_1), \dots, \kappa(t_{n-1})] \otimes \kappa(t_n), t_{n+1}] \\ &= [\kappa(t_1), \dots, \kappa(t_n)] \otimes \kappa(t_{n+1}). \end{aligned}$$

If G' is nilpotent with $cl(G') = c$, then for any $t_1, \dots, t_{c+2} \in G \otimes G$ we have $[t_1, \dots, t_{c+2}] = [\kappa(t_1), \dots, \kappa(t_{c+1})] \otimes \kappa(t_{c+2}) = 1_G \otimes \kappa(t_{c+2}) = 1_{\otimes}$. Thus $cl(G \otimes G) \leq cl(G') + 1$.

On the other hand, for any $n < cl(G')$ there exist elements g_1, \dots, g_n in G' such that $[g_1, \dots, g_n] \neq 1$. Since κ is onto G' there exist elements $t_1, \dots, t_n \in G \otimes G$ with $\kappa(t_i) = g_i$, $i = 1, \dots, n$. Thus $\kappa([t_1, \dots, t_n]) = [\kappa(t_1), \dots, \kappa(t_n)] = [g_1, \dots, g_n] \neq 1$. Therefore $[t_1, \dots, t_n] \neq 1_{\otimes}$ and $cl(G \otimes G) \geq cl(G')$. \square

The following familiar identities for 2-Engel groups, i.e. groups satisfying the law $[x, y, y]$, are stated here without proof and will be used without further reference (see e.g. [11]). Note that any nilpotent group of class 2 is a 2-Engel group.

Lemma 2.3. *Let G be a 2-Engel group. Then:*

$$(2.3.1) \quad [x, y, z, w] = [x, y, z]^3 = 1 \text{ for any } x, y, z, w \in G;$$

$$(2.3.2) \quad [x, y, z] = [z, x, y] = [x, z, y]^{-1} \text{ for any } x, y, z \in G;$$

$$(2.3.3) \quad \langle x, y, [v, z] \rangle \text{ is nilpotent of class 2 for any } x, y, v, z \in G;$$

$$(2.3.4) \quad [x, y^n] = [x^n, y] = [x, y]^n \text{ and } (xy)^n = x^n y^n [y, x]^{\binom{n}{2}} \text{ for } x, y \in G \text{ and } n \in \mathbb{Z}.$$

We give now several lemmas establishing various identities in the nonabelian tensor square of a 2-Engel group. The first one is a restatement of Lemma 3.4 in [3].

Lemma 2.4. *Let G be a nilpotent group of class at most 2, then for $x, y \in G$ and $n, m \in \mathbb{Z}$*

$$x^n \otimes y^m = (x \otimes y)^{nm} (y \otimes [x, y])^{n \binom{m}{2}} (x \otimes [x, y])^{m \binom{n}{2}}.$$

The next lemma will be stated without proof. It is an easy consequence of the preceding results and (11) of Proposition 3 in [4] which we restate in (2.6.3) below.

Lemma 2.5. *Let G be a 2-Engel group. For $x, y, z, v \in G$ and $i, j, k, l \in \mathbb{Z}$ we have*

$$(2.5.1) \quad [x^i \otimes v^j, y^k \otimes z^l] = [x \otimes v, y \otimes z]^{ijkl};$$

$$(2.5.2) \quad [x, v]^i \otimes [y, z] = [x, v] \otimes [y, z]^i = ([x, v] \otimes [y, z])^i;$$

$$(2.5.3) \quad [x, v] \otimes [y, z] = ([y, z] \otimes [x, v])^{-1} = ([v, x] \otimes [y, z])^{-1} = ([x, v] \otimes [z, y])^{-1}.$$

Lemma 2.6. *Let G be a 2-Engel group. Then, for any $v, x, y, z \in G$ the following hold.*

$$(2.6.1) \quad [x \otimes v, y \otimes z] = ([x, v, y] \otimes z)(y \otimes [x, v, z]);$$

$$(2.6.2) \quad ([x, v, y] \otimes z)(y \otimes [x, v, z])([y, z, x] \otimes v)(x \otimes [y, z, v]) = 1_{\otimes}.$$

Proof. We observe that $G \otimes G$ is nilpotent of class at most 2. By (2.3.3) and Proposition 3.1 in [3] we have $[[x, v] \otimes y, [x, v] \otimes z] = 1_{\otimes}$. Now

$$(2.6.3) \quad [x \otimes v, y \otimes z] = [x, v] \otimes [y, z].$$

implies $[x, v, y] \otimes [x, v, z] = 1_{\otimes}$. Thus

$$\begin{aligned} [x \otimes v, y \otimes z] &= (x^{\otimes v} (y \otimes z))(y \otimes z)^{-1} \\ &= ([x, v] (y \otimes z))(y \otimes z)^{-1} \\ &= ([x, v, y] y \otimes [x, v, z] z)(y \otimes z)^{-1} \\ &= ([x, v, y] \otimes [x, v, z])([x, v, y] \otimes z)(y \otimes [x, v, z])(y \otimes z)(y \otimes z)^{-1} \\ &= ([x, v, y] \otimes z)(y \otimes [x, v, z]), \end{aligned}$$

and (2.6.1) follows. By relabeling in (2.6.1) we obtain $[y \otimes z, x \otimes v] = ([y, z, x] \otimes v)(x \otimes [y, z, v])$. This together with (2.6.1) and (2.5.3) yields (2.6.2). \square

The following corollary is a consequence of Lemma 2.3 and the lemma above.

Corollary 2.7. *Let G be a 2-Engel group. Then, for any $x, y, z \in G$, the following hold:*

$$(2.7.1) \quad [x \otimes y, y \otimes z] = (y \otimes [x, y, z]) = ([z, y, x] \otimes y);$$

$$(2.7.2) \quad (x \otimes [x, y, z])^{-1} = ([x, y, z] \otimes x);$$

$$(2.7.3) \quad [x \otimes y, y \otimes z]^3 = 1_{\otimes}.$$

Proof. Setting $v = y$ in (2.6.1) gives $[x \otimes y, y \otimes z] = y \otimes [x, y, z]$. Substituting z for x , y for v and z , and x for y in (2.6.1) yields $[x, y] \otimes [x, y] = [z, y, x] \otimes y$. But $[z, y] \otimes [x, y] = [x, y] \otimes [y, z]$ by (2.5.3). We conclude that (2.7.1) holds.

Using (2.3.2) and then interchanging x and y in (2.7.1) yields (2.7.2). To show (2.7.3), we observe that $[x, y, z]^3 = 1$ by (2.3.2). Hence by expansion, (2.7.1) and (2.6.3), we obtain $1_{\otimes} = y \otimes [x, y, z]^3 = (y \otimes [x, y, z])^3 = ([x, y] \otimes [y, z])^3$. \square

Proposition 2.8. *For a 2-Engel group G the defining relations of $G \otimes G$ reduce to*

$$(2.8.1) \quad xy \otimes z = (x \otimes [y, z])(y \otimes z)(x \otimes z),$$

$$(2.8.2) \quad x \otimes yz = ([z, x] \otimes y)(x \otimes y)(x \otimes z), \text{ for all } x, y, z \in G.$$

Proof. Again observe that G has class 3 and $G \otimes G$ has class not exceeding 2. By Lemma 2.3 and the defining relations of the nonabelian tensor square in general we obtain

$$\begin{aligned} yx \otimes z &= {}^y(x \otimes z)(y \otimes z) = ([y, x]x \otimes [y, z]z)(y \otimes z) \\ &= (x \otimes [y, x, z][y, z]z)([y, x] \otimes [y, z]z)(y \otimes z) \\ &= (x \otimes [y, x, z])(x \otimes [y, z])(x \otimes z)([y, z, x] \otimes z)([y, x] \otimes [y, z])([y, x] \otimes z)(y \otimes z). \end{aligned}$$

Note that all terms of the form $g \otimes h$ where g or h is a commutator are in the center of $G \otimes G$, and that $(x \otimes z)(y \otimes z) = [x \otimes z, y \otimes z](y \otimes z)(x \otimes z)$. Using the above information together with (2.6.3) and (2.7.1) we obtain after cancellation

$$yx \otimes z = (xy \otimes [y, x, z])(x \otimes [y, z])([y, x] \otimes z)(y \otimes z)(x \otimes z).$$

On the other hand,

$$\begin{aligned} [y, x]xy \otimes z &= {}^{[y, x]}(xy \otimes z)([y, x] \otimes z) = (xy \otimes [y, x, z]z)([y, x] \otimes z) \\ &= (xy \otimes [y, x, z])(xy \otimes z)([y, x] \otimes z). \end{aligned}$$

By observing $yx = [y, x]xy$ and by equating the two expansions we arrive at $xy \otimes z = (x \otimes [y, z])(y \otimes z)(x \otimes z)$, and (2.8.1) holds.

Similarly, one obtains (2.8.2) by expanding $x \otimes zy$ and comparing it to the expansion of $x \otimes [z, y]yz$. \square

We are ready for the last proposition of this section.

Proposition 2.9. *If G is a 2-Engel group, then the following relations hold for all elements x, y, z in G .*

$$(2.9.1) \quad (z \otimes [y, x])(y \otimes [x, z])(x \otimes [z, y]) = [y \otimes z, x \otimes z][y \otimes z, x \otimes y][x \otimes z, x \otimes y];$$

$$(2.9.2) \quad ([x, y] \otimes z)^{-1} = ([y, x] \otimes z) = (z \otimes [y, x])^{-1} = (z \otimes [x, y]).$$

Proof. By Proposition 3 in [4] and expansion we have

$$\begin{aligned} ([x, y] \otimes z)^{-1} &= [x, y]([y, x] \otimes z) = ([y, x] \otimes [x, y, z]z) \\ &= ([y, x] \otimes [x, y, z])([y, x] \otimes z) = ([y, x] \otimes [y, z, x])([y, x] \otimes z) \\ &= ([y \otimes x, [y, z] \otimes x])([y, x] \otimes z) = ([y, x] \otimes z). \end{aligned}$$

Thus the first equality in (2.9.2) holds. Similarly,

$$(z \otimes [x, y])^{-1} = [x, y](z \otimes [y, x]) = (z \otimes [y, x]),$$

and we have the third equality in (2.9.2).

Now consider the element $xy \otimes z \in G \otimes G$. Using Proposition 2.8, expand this element first as

$$xy \otimes z = (x \otimes [y, z])(y \otimes z)(x \otimes z) = (x \otimes [y, z])[y \otimes z, x \otimes z](x \otimes z)(y \otimes z),$$

and secondly as

$$\begin{aligned} [x, y]yx \otimes z &= ([x, y] \otimes [yx, z])([x, y] \otimes z)(yx \otimes z) \\ &= [x \otimes y, y \otimes z][x \otimes y, x \otimes z]([x, y] \otimes z)(y \otimes [x, z])(x \otimes z)(y \otimes z). \end{aligned}$$

By equating the terms we obtain

$$(2.9.3) \quad ([x, y] \otimes z)(y \otimes [x, z])(x \otimes [z, y]) = [y \otimes z, x \otimes z][x \otimes y, z \otimes y][x \otimes y, z \otimes x].$$

Similarly, expanding $x \otimes yz$ and comparing to the expansion of $x \otimes [y, z]zy$ yields

$$(2.9.4) \quad (x \otimes [y, z])([y, x] \otimes z)([x, z] \otimes y) = [x \otimes y, x \otimes z][y \otimes z, z \otimes x][y \otimes z, y \otimes x].$$

Notice that the right side of (2.9.3) is the inverse of the right side of (2.9.4) by Lemma 2.4. By comparing (2.9.3) and (2.9.4) we obtain $[x, z] \otimes y = y \otimes [x, z]$. Relabeling leads to the middle equality in (2.9.2). Now (2.9.2) together with (2.9.3) imply (2.9.1). \square

In conclusion we want to mention that (2.9.1) came to us somewhat as a surprise. Under the mapping $\kappa : G \otimes G \rightarrow G'$ each term on the right side of (2.9.1) is equal to the identity.

We expected that this was also the case in the preimage $G \otimes G$. As a consequence, $G \otimes G$ would have been abelian. But as it turned out, (2.9.1) holds in $G \otimes G$ in a nontrivial way resulting in $G \otimes G$ being nilpotent of class 2.

3. Computing the tensor square. In this section we construct the nonabelian tensor square of \mathcal{E} , the free 2-Engel group of rank 3. We first give an explicit presentation of \mathcal{E} and the multiplication and conjugation formulas for this group.

Example 3.1. Let $T = F_3/\gamma_4(F_3) = \langle y_1, y_2, y_3 \rangle$ be the free nilpotent group of class 3 and rank 3, where F_3 is the free group of rank 3. By the Jacobi identity we obtain $\gamma_3(T) = \langle [y_1, y_2, y_3], [y_3, y_1, y_2], [y_i, y_j, y_j], i, j = 1, 2, 3 \rangle$. Consider

$$N = \langle [y_1, y_2, y_3]^3, [y_1, y_2, y_3][y_3, y_1, y_2]^{-1}, [y_i, y_j, y_j], i, j = 1, 2, 3 \rangle.$$

Now $N \subseteq \gamma_3(T) \subseteq Z(T)$, thus $N \triangleleft T$. Therefore $\mathcal{E} = T/N = \langle a, b, c \rangle$, where $a = y_1N$, $b = y_2N$ and $c = y_3N$.

From the construction of \mathcal{E} it follows that every $g \in \mathcal{E}$ has a unique presentation as

$$(3.1.1) \quad g = a^\alpha b^\beta c^\gamma [a, b]^\rho [a, c]^\sigma [b, c]^\tau [a, b, c]^\lambda,$$

where $\alpha, \beta, \gamma, \rho, \sigma, \tau$ are integers and λ is an integer modulo 3.

Let $h \in \mathcal{E}$ with $h = a^{\alpha'} b^{\beta'} c^{\gamma'} [a, b]^{\rho'} [a, c]^{\sigma'} [b, c]^{\tau'} [a, b, c]^{\lambda'}$. Then we have the following multiplication and conjugation formulas in \mathcal{E} :

$$(3.1.2) \quad gh = a^{\alpha+\alpha'} b^{\beta+\beta'} c^{\gamma+\gamma'} [a, b]^{\rho^*} [a, c]^{\sigma^*} [b, c]^{\tau^*} [a, b, c]^{\lambda^*},$$

where $\rho^* = \rho + \rho' - \alpha'\beta$, $\sigma^* = \sigma + \sigma' - \alpha'\gamma$, $\tau^* = \tau + \tau' - \beta'\gamma$, and $\lambda^* \equiv \lambda + \lambda' + \rho\gamma' - \sigma\beta' + \tau\alpha' - \alpha'\beta\gamma' - \alpha'\beta\gamma + \alpha'\beta'\gamma \pmod{3}$;

$$(3.1.3) \quad {}^g h = a^{\alpha'} b^{\beta'} c^{\gamma'} [a, b]^{\bar{\rho}} [a, c]^{\bar{\sigma}} [b, c]^{\bar{\tau}} [a, b, c]^{\bar{\lambda}},$$

where $\bar{\rho} = \rho' + \alpha\beta' - \alpha'\beta$, $\bar{\sigma} = \sigma' + \alpha\gamma' - \alpha'\gamma$, $\bar{\tau} = \tau' + \beta\gamma' - \beta'\gamma$, and $\bar{\lambda} \equiv \lambda' + \rho\gamma' - \rho'\gamma - \sigma\beta' + \sigma'\beta + \tau\alpha' - \tau'\alpha - \alpha'\beta\gamma' + \alpha'\beta'\gamma - \alpha'\beta\gamma + \alpha\beta'\gamma + \alpha\beta'\gamma' - \alpha\beta\gamma'$.
□

In our next example we will construct a group L which we will show in Theorem 3.6 is isomorphic to the nonabelian tensor square of \mathcal{E} .

Example 3.2. We construct first a group S as a suitable quotient of a free group. Let F_6 be the free group of rank 6 and set $\mathcal{H}_6 = F_6/\gamma_3(F_6) = \langle y_1, \dots, y_6 \rangle$, the free nilpotent

group of class 2 and rank 6. Let $N = \langle [y_1, y_6], [y_2, y_4], [y_3, y_5], [y_5, y_6]^3, [y_5, y_4]^3, [y_6, y_4]^3 \rangle$. Since $N \leq \mathcal{H}'_6 \leq Z(\mathcal{H}_6)$, we have $N \triangleleft \mathcal{H}_6$. Set $S = \mathcal{H}_6/N = \langle x_1, \dots, x_6 \rangle$, where $x_i = y_i N$, $i = 1, \dots, 6$.

It follows that $g \in S$ has a unique presentation as $g = z_1^{\mu_1} z_2^{\mu_2} z_3^{\mu_3} x_1^{\kappa_1} \dots x_6^{\kappa_6}$, where $\kappa_1, \dots, \kappa_6$ are integers, μ_1, μ_2, μ_3 integers modulo 3, and $z_1 = [x_5, x_6]$, $z_2 = [x_5, x_4]$, $z_3 = [x_6, x_4]$. Let $h \in S$ with $h = z_1^{\mu'_1} z_2^{\mu'_2} z_3^{\mu'_3} x_1^{\kappa'_1} \dots x_6^{\kappa'_6}$. Then

$$(3.2.1) \quad gh = z_1^{\mu_1^*} z_2^{\mu_2^*} z_3^{\mu_3^*} x_1^{\kappa_1^*} \dots x_6^{\kappa_6^*}$$

with

$$\begin{aligned} \kappa_i^* &= \kappa_i + \kappa'_i, \quad i = 1, \dots, 6, \\ \mu_1^* &\equiv \mu_1 + \mu'_1 + \kappa_3 \kappa'_1 + \kappa_6 \kappa'_3 - \kappa_6 \kappa'_5 - \kappa_5 \kappa'_1 \pmod{3}, \\ \mu_2^* &\equiv \mu_2 + \mu'_2 - \kappa_5 \kappa'_2 + \kappa_3 \kappa'_2 + \kappa_4 \kappa'_3 + \kappa_5 \kappa'_4 \pmod{3}, \\ \mu_3^* &\equiv \mu_3 + \mu'_3 + \kappa_4 \kappa'_1 - \kappa_6 \kappa'_2 - \kappa_2 \kappa'_1 + \kappa_6 \kappa'_4 \pmod{3}. \end{aligned}$$

Let $A = \langle u_1, \dots, u_{11} \rangle$ be a free abelian group of rank 11. Then set $L = A \times S$. \square

Our next proposition contains an expansion formula for $g \otimes h$, $g, h \in \mathcal{E}$.

Proposition 3.3. *Let \mathcal{E} be the free 2-Engel group of rank 3 and $g, h \in \mathcal{E}$ with $g = a^\alpha b^\beta c^\gamma [a, b]^\rho [a, c]^\sigma [b, c]^\tau [a, b, c]^\lambda$ and $h = a^{\alpha'} b^{\beta'} c^{\gamma'} [a, b]^{\rho'} [a, c]^{\sigma'} [b, c]^{\tau'} [a, b, c]^{\lambda'}$. Then*

$$(3.3.1) \quad \left\{ \begin{array}{l} g \otimes h = (a \otimes a)^{\alpha\alpha'} (b \otimes b)^{\beta\beta'} (c \otimes c)^{\gamma\gamma'} \cdot \\ \quad (a \otimes [a, b])^{\nu_4} (a \otimes [a, c])^{\nu_5} (b \otimes [a, b])^{\nu_6} \cdot \\ \quad (b \otimes [b, c])^{\nu_7} (c \otimes [a, c])^{\nu_8} (c \otimes [b, c])^{\nu_9} \cdot \\ \quad (a \otimes [b, c])^{\nu_{10}} (b \otimes [a, c])^{\nu_{11}} \cdot \\ \quad ([a, b] \otimes [a, c])^{\mu_1} ([a, b] \otimes [b, c])^{\mu_2} ([a, c] \otimes [b, c])^{\mu_3} \cdot \\ \quad (c \otimes a)^{\alpha'\gamma} (c \otimes b)^{\beta'\gamma} (b \otimes a)^{\alpha'\beta} (b \otimes c)^{\beta\gamma'} (a \otimes b)^{\alpha\beta'} (a \otimes c)^{\alpha\gamma'}, \end{array} \right.$$

where

$$(3.3.2) \quad \left\{ \begin{array}{l} \nu_4 = \alpha\rho' - \alpha'\rho + \beta' \binom{\alpha}{2} - \beta \binom{\alpha'}{2} - \alpha\alpha'\beta + \alpha\alpha'\beta', \\ \nu_5 = \alpha\sigma' - \alpha'\sigma + \binom{\alpha}{2}\gamma' - \binom{\alpha'}{2}\gamma - \alpha\alpha'\gamma + \alpha\alpha'\gamma', \\ \nu_6 = \alpha \binom{\beta'}{2} + \beta\rho' - \beta'\rho - \alpha' \binom{\beta}{2}, \\ \nu_7 = \beta\tau' - \beta'\tau + \binom{\beta}{2}\gamma' - \binom{\beta'}{2}\gamma - \beta\beta'\gamma + \beta\beta'\gamma', \\ \nu_8 = \alpha \binom{\gamma'}{2} + \gamma\sigma' - \gamma'\sigma - \alpha' \binom{\gamma}{2}, \\ \nu_9 = \beta \binom{\gamma'}{2} + \gamma\tau' - \gamma'\tau - \beta' \binom{\gamma}{2}, \\ \nu_{10} = \alpha\tau' - \alpha'\tau + \rho\gamma' - \rho'\gamma + \alpha\beta\gamma' - \alpha\beta'\gamma + \alpha'\beta\gamma' - \alpha'\beta'\gamma, \\ \nu_{11} = \beta\sigma' - \beta'\sigma - \rho\gamma' + \rho'\gamma + \alpha\beta'\gamma' - \alpha'\beta\gamma, \end{array} \right.$$

and

$$(3.3.3) \quad \left\{ \begin{array}{l} \mu_1 \equiv \alpha\lambda' - \alpha'\lambda + \binom{\alpha'}{2}\tau - \rho\gamma' + \rho\sigma' + \rho'\gamma - \rho'\sigma - \binom{\alpha}{2}\tau' + \alpha\alpha'\tau - \alpha\alpha'\tau' + \\ \quad \alpha\beta\sigma' - \alpha\beta'\sigma + \alpha\rho\gamma' - \alpha\rho'\gamma + \alpha'\beta\sigma' - \alpha'\beta'\sigma + \alpha'\rho\gamma' - \alpha'\rho'\gamma - \beta \binom{\alpha'}{2}\gamma' + \\ \quad \beta' \binom{\alpha'}{2}\gamma - \beta \binom{\alpha}{2}\gamma' + \beta' \binom{\alpha}{2}\gamma - \alpha\alpha'\beta\gamma - \alpha\alpha'\beta'\gamma + \alpha\alpha'\beta'\gamma + \alpha\alpha'\beta'\gamma' \pmod{3}, \\ \mu_2 \equiv \beta\lambda' - \beta'\lambda + \binom{\beta}{2}\sigma' - \rho\gamma' + \rho\tau' + \rho'\gamma - \rho'\tau - \binom{\beta'}{2}\sigma - \alpha\beta'\tau' + \\ \quad \alpha'\beta\tau - \alpha' \binom{\beta}{2}\gamma - \beta\beta'\sigma + \beta\beta'\sigma' + \beta\rho\gamma' - \beta\rho'\gamma + \beta'\rho\gamma' - \beta'\rho'\gamma + \\ \quad \alpha \binom{\beta'}{2}\gamma' + \alpha'\beta\beta'\gamma \pmod{3}, \\ \mu_3 \equiv \binom{\gamma'}{2}\rho - \rho\gamma' + \rho'\gamma + \gamma\lambda' - \gamma'\lambda + \sigma\tau' - \sigma'\tau - \binom{\gamma}{2}\rho' - \\ \quad \alpha\gamma'\tau' + \alpha'\gamma\tau + \beta\gamma'\sigma' - \beta'\gamma\sigma + \rho\gamma\gamma' - \rho'\gamma\gamma' \pmod{3}. \end{array} \right.$$

Proof. The formula follows by repeatedly applying the expansion and collection rules for elements in $\mathcal{E} \otimes \mathcal{E}$ and \mathcal{E} as given by Lemmas 2.3, 2.4, and 2.5, Corollary 2.7, as well as Propositions 2.8 and 2.9. Thus $g \otimes h$ in $\mathcal{E} \otimes \mathcal{E}$ can be expressed as a product of the 20 terms listed in the expansion formula (3.3.1). For the first eleven and the last six terms this expansion and collection was completed by hand. A program written in the GAP programming language [12] was used to verify the hand calculations and to calculate the exponents of the terms $[a, b] \otimes [a, c]$, $[a, b] \otimes [b, c]$, and $[a, c] \otimes [b, c]$ in (3.3.3). The identity (2.9.1) allowed us to eliminate any term of the form $c \otimes [a, b]$ from the expansion. For the collection of elements we observed in addition that only the last 6 terms in the expansion, namely $c \otimes a$, $c \otimes b$, $b \otimes a$, $b \otimes c$, $a \otimes b$ and $a \otimes c$, are not in the center of $\mathcal{E} \otimes \mathcal{E}$. They were left in the order as they appeared in the expansion. \square

The following corollary is an immediate consequence of the above and (2.6.3).

Corollary 3.4. *The elements $t \in \mathcal{E} \otimes \mathcal{E}$ can be represented in the form $t = U \cdot V_1 \cdot V_2 \cdot W \cdot X$, where*

$$\begin{aligned}
U &= (a \otimes a)^{\eta_1} (b \otimes b)^{\eta_2} (c \otimes c)^{\eta_3}, \\
V_1 &= (a \otimes [a, b])^{\eta_4} (a \otimes [a, c])^{\eta_5} (b \otimes [a, b])^{\eta_6} (b \otimes [b, c])^{\eta_7} (c \otimes [a, c])^{\eta_8} (c \otimes [b, c])^{\eta_9}, \\
V_2 &= (a \otimes [b, c])^{\eta_{10}} (b \otimes [a, c])^{\eta_{11}}, \\
W &= ([a, b] \otimes [a, c])^{\zeta_1} ([a, b] \otimes [b, c])^{\zeta_2} ([a, c] \otimes [b, c])^{\zeta_3}, \\
X &= (c \otimes a)^{\theta_1} (c \otimes b)^{\theta_2} (b \otimes a)^{\theta_3} (b \otimes c)^{\theta_4} (a \otimes b)^{\theta_5} (a \otimes c)^{\theta_6},
\end{aligned}$$

for integers η_1, \dots, η_{11} , $\theta_1, \dots, \theta_6$, and for $\zeta_1, \zeta_2, \zeta_3$ integers modulo 3.

For easier reference, the next lemma gives two binomial identities without proof.

Lemma 3.5. *For integers n, m we have*

$$(3.5.1) \quad \binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn,$$

$$(3.5.2) \quad \binom{m}{2} \equiv m - m^2 \pmod{3}.$$

Now we prove our main result.

Theorem 3.6. *Let \mathcal{E} be the free 2-Engel group of rank 3. Then $\mathcal{E} \otimes \mathcal{E} \cong L$, where L is the group of Example 3.2.*

Proof. We first define a mapping $\Phi : \mathcal{E} \times \mathcal{E} \rightarrow L$ and show that it is a crossed pairing. We start by defining Φ for the pairs (g, h) , where $g \otimes h$ is one of the 20 terms appearing on the right side of the expansion formula in Proposition 3.3. Set

$$\begin{aligned}
(3.6.1) \quad & \begin{aligned}
\Phi(a, a) &= u_1, & \Phi(b, b) &= u_2, & \Phi(c, c) &= u_3, \\
\Phi(a, [a, b]) &= u_4, & \Phi(a, [a, c]) &= u_5, & \Phi(b, [a, b]) &= u_6, \\
\Phi(b, [b, c]) &= u_7, & \Phi(c, [a, c]) &= u_8, & \Phi(c, [b, c]) &= u_9, \\
\Phi(a, [b, c]) &= u_{10}, & \Phi(b, [a, c]) &= u_{11}, & & \\
\Phi([a, b], [a, c]) &= z_1, & \Phi([a, b], [b, c]) &= z_2, & \Phi([a, c], [b, c]) &= z_3, \\
\Phi(c, a) &= x_1, & \Phi(c, b) &= x_2, & \Phi(b, a) &= x_3, \\
\Phi(b, c) &= x_4, & \Phi(a, b) &= x_5, & \Phi(a, c) &= x_6.
\end{aligned}
\end{aligned}$$

We represent the elements $l \in L$ as 20-tuples, listing the exponents of the respective generators as coordinates as follows:

$$(3.6.2) \quad l = (\exp(u_1), \dots, \exp(u_{11}), \exp(z_1), \dots, \exp(z_3), \exp(x_1), \dots, \exp(x_6)),$$

where $\exp(u_1), \dots, \exp(u_{11})$ and $\exp(x_1), \dots, \exp(x_6)$ are integers and $\exp(z_1), \exp(z_2)$ and $\exp(z_3)$ are integers modulo 3. We denote by $\bar{0}$ the identity element in L . Set

$$(3.6.3) \quad \begin{cases} g = a^\alpha b^\beta c^\gamma [a, b]^\rho [a, c]^\sigma [b, c]^\tau [a, b, c]^\lambda, \\ h = a^{\alpha'} b^{\beta'} c^{\gamma'} [a, b]^{\rho'} [a, c]^{\sigma'} [b, c]^{\tau'} [a, b, c]^{\lambda'}, \\ g' = h' = a^{\alpha''} b^{\beta''} c^{\gamma''} [a, b]^{\rho''} [b, c]^{\tau''} [a, b, c]^{\lambda''}. \end{cases}$$

Using (3.3.1), we define now $\Phi(g, h)$, $g, h \in \mathcal{E}$, as follows:

$$(3.6.4) \quad \Phi(g, h) = (\alpha\alpha', \beta\beta', \gamma\gamma', \nu_4, \dots, \nu_{11}, \mu_1, \mu_2, \mu_3, \alpha'\gamma, \beta'\gamma, \alpha'\beta, \beta\gamma', \alpha\beta', \alpha\gamma'),$$

where ν_4, \dots, ν_{11} are the integers given in (3.3.2) and, μ_1, μ_2, μ_3 the integers modulo 3 given in (3.3.3). Since $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \rho, \rho', \sigma, \sigma', \tau, \tau'$ are unique integers and λ, λ' are unique integers modulo 3, the mapping $\Phi : \mathcal{E} \times \mathcal{E} \rightarrow L$ given by (3.6.4) is well-defined.

Next we need to verify that Φ is a crossed pairing, i.e. (1.1.1) and (1.1.2) are satisfied. Through computer calculations, we obtain

$$(3.6.5) \quad \Phi(gg', h)(\Phi({}^g g', {}^g h) \cdot \Phi(g, h))^{-1} = (0, 0, 0, \bar{\nu}_4, \dots, \bar{\nu}_9, 0, 0; \bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, 0, \dots, 0),$$

where

$$\begin{aligned} \bar{\nu}_4 &= -\beta' \binom{\alpha''}{2} + \beta' \binom{\alpha + \alpha''}{2} - \beta' \binom{\alpha}{2} - \alpha\alpha''\beta', \\ \bar{\nu}_5 &= -\binom{\alpha''}{2}\gamma' + \binom{\alpha + \alpha''}{2}\gamma' - \binom{\alpha}{2}\gamma' - \alpha\alpha''\gamma', \\ \bar{\nu}_6 &= \alpha' \binom{\beta''}{2} - \alpha' \binom{\beta + \beta''}{2} + \alpha' \binom{\beta}{2} + \alpha'\beta\beta'', \\ \bar{\nu}_7 &= -\binom{\beta''}{2}\gamma' + \binom{\beta + \beta''}{2}\gamma' - \binom{\beta}{2}\gamma' - \beta\beta''\gamma', \\ \bar{\nu}_8 &= \alpha' \binom{\gamma''}{2} - \alpha' \binom{\gamma + \gamma''}{2} + \alpha' \binom{\gamma}{2} + \alpha'\gamma\gamma'', \\ \bar{\nu}_9 &= \beta' \binom{\gamma''}{2} - \beta' \binom{\gamma + \gamma''}{2} + \beta' \binom{\gamma}{2} + \beta'\gamma\gamma'', \end{aligned}$$

and

$$\begin{aligned}
\mu_1 &\equiv \binom{\alpha''}{2}\tau' - \binom{\alpha + \alpha''}{2}\tau' + \binom{\alpha}{2}\tau' + \alpha\alpha''\tau' - \alpha\beta'\gamma'' + \alpha\beta''\gamma' + \alpha'\beta\gamma'' + \\
&\quad \beta\binom{\alpha''}{2}\gamma' - \beta\binom{\alpha'}{2}\gamma'' - \beta\binom{\alpha + \alpha''}{2}\gamma' + \beta\binom{\alpha}{2}\gamma' - \beta'\binom{\alpha''}{2}\gamma - \beta'\binom{\alpha''}{2}\gamma'' + \\
&\quad \beta'\binom{\alpha + \alpha''}{2}\gamma + \beta'\binom{\alpha + \alpha''}{2}\gamma'' - \beta'\binom{\alpha}{2}\gamma + \beta''\binom{\alpha''}{2}\gamma' - \beta''\binom{\alpha + \alpha''}{2}\gamma' + \alpha\alpha\beta'\gamma'' - \\
&\quad \alpha\alpha\beta''\gamma' + 3\alpha\alpha'\beta'\gamma'' - 3\alpha\alpha'\beta''\gamma' - 2\alpha\alpha''\beta\gamma' - \alpha\alpha''\beta'\gamma + 2\alpha\alpha''\beta'\gamma'' - 2\alpha\alpha''\beta''\gamma' - \\
&\quad \alpha'\alpha'\beta\gamma'' - 3\alpha'\alpha''\beta\gamma'' \pmod{3} \\
\mu_2 &\equiv -\binom{\beta''}{2}\sigma' + \binom{\beta + \beta''}{2}\sigma' - \binom{\beta}{2}\sigma' - \alpha\beta'\gamma'' + \alpha\beta''\gamma' - \alpha\binom{\beta''}{2}\gamma' + \\
&\quad \alpha\binom{\beta'}{2}\gamma'' + \alpha'\beta\gamma'' + \alpha'\binom{\beta''}{2}\gamma + \alpha'\binom{\beta''}{2}\gamma'' - \alpha'\binom{\beta + \beta''}{2}\gamma - \alpha'\binom{\beta + \beta''}{2}\gamma'' + \\
&\quad \alpha'\binom{\beta}{2}\gamma - \beta\beta''\sigma' + 3\alpha\beta\beta'\gamma'' - 3\alpha\beta\beta''\gamma' + \alpha\beta'\beta'\gamma'' - 3\alpha\beta'\beta''\gamma' + \\
&\quad 3\alpha\beta'\beta''\gamma'' - \alpha\beta''\beta''\gamma' - \alpha'\beta\beta\gamma'' + \alpha'\beta\beta''\gamma - 2\alpha'\beta\beta''\gamma'' \pmod{3}, \\
\mu_3 &\equiv \binom{\gamma''}{2}\rho' - \binom{\gamma + \gamma''}{2}\rho' + \binom{\gamma}{2}\rho' + \alpha\beta'\binom{\gamma''}{2} - \alpha\beta'\gamma'' - \alpha\beta''\binom{\gamma'}{2} + \\
&\quad \alpha\beta''\gamma' - \alpha'\beta\binom{\gamma''}{2} + \alpha'\beta\gamma'' + \rho'\gamma\gamma'' + \alpha\beta'\gamma''\gamma'' - \alpha\beta''\gamma'\gamma' - \alpha'\beta\gamma''\gamma'' \pmod{3}.
\end{aligned}$$

The exponents of the terms $\Phi(a, [a, b]), \dots, \Phi(c, [b, c])$, i.e. $\bar{\nu}_4, \dots, \bar{\nu}_9$ above, are identical to zero by (3.5.1). The exponents of $\Phi([a, b], [a, c]), \Phi([a, b], [b, c])$ and $\Phi([a, c], [b, c])$, i.e. $\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3$ above, are identical to zero modulo 3 by (3.5.2). Hence (3.6.5) becomes

$$\Phi(gg', h)(\Phi(gg', {}^g h)\Phi(g, h))^{-1} = \bar{0},$$

and (1.1.1) holds.

It should be mentioned here again that the exponents of the first eleven and the last six components in this verification were shown to be equal to zero by hand calculations. The GAP program was used to check these hand calculations and to complete the verification that Φ is a crossed pairing by showing the exponents of the remaining components were congruent zero modulo 3.

A similar calculation together with Lemma 3.5 gives $\Phi(g, hh')(\Phi(g, h)\Phi({}^h g, {}^h h'))^{-1} = \bar{0}$. The details are omitted. Thus (1.1.2) is satisfied and Φ is a crossed pairing. By Proposition 1.2 it follows now that there exists a homomorphism Φ^* of $\mathcal{E} \otimes \mathcal{E}$ into L .

It remains to be shown that Φ^* is one-one and onto. Since $\Phi^*(g \otimes h) = \Phi(g, h)$, it follows by Corollary 3.4 and (3.6.1) that Φ^* is onto L . We construct now an epimorphism $\psi^* : L \rightarrow \mathcal{E} \otimes \mathcal{E}$ as follows. For $u \in \{u_1, \dots, u_{11}, z_1, z_2, z_3, x_1, \dots, x_6\}$, the set of generators of L , we define $\psi(u) = \Phi^{-1}(u) = (v, w) \in \mathcal{E} \times \mathcal{E}$ as given by (3.6.1). Since $\Phi^*(g \otimes h) = \Phi(g, h)$ we obtain a mapping ψ^* of the generators of L to the generators of $\mathcal{E} \otimes \mathcal{E}$ as given

by Corollary 3.4. The mapping ψ^* defined on the generators of L can be extended to a mapping of L into $\mathcal{E} \otimes \mathcal{E}$ as follows. For $l \in L$ as given by (3.6.2), set $\psi^*(l) = t \in \mathcal{E} \otimes \mathcal{E}$, where t is defined as in Corollary 3.4 with

$$(3.6.6) \quad \begin{cases} \eta_i = \exp(u_i), & i = 1, \dots, 11, \\ \zeta_i = \exp(z_i), & i = 1, 2, 3, \\ \theta_i = \exp(x_i), & i = 1, \dots, 6. \end{cases}$$

Observing Lemma 2.5, (2.6.3), and

$$(3.6.7) \quad \begin{cases} \psi^*(z_1) = \psi^*([x_5, x_6]) = [\psi^*(x_5), \psi^*(x_6)] = [a \otimes b, a \otimes c], \\ \psi^*(z_2) = \psi^*([x_5, x_4]) = [\psi^*(x_5), \psi^*(x_4)] = [a \otimes b, b \otimes c], \\ \psi^*(z_3) = \psi^*([x_6, x_4]) = [\psi^*(x_6), \psi^*(x_4)] = [a \otimes c, b \otimes c], \end{cases}$$

it can be verified that $\psi^* : L \rightarrow \mathcal{E} \otimes \mathcal{E}$ is a homomorphism of L onto $\mathcal{E} \otimes \mathcal{E}$. It follows that $\Phi^*\psi^*$ is an endomorphism of L onto L . Since L is a finitely generated nilpotent group, we conclude that $\Phi^*\psi^*$ is an automorphism of L (see e.g. (6.1.11) in [11]). It follows that Φ^* is one-one. \square

Corollary 3.7. *Let $B(3, 3)$ be the Burnside group of exponent 3 and rank 3. Then $B(3, 3) \otimes B(3, 3) \cong L/L^3$, where L is the group of Example 3.2.*

Proof. For brevity set $B = B(3, 3)$. We observe $B \cong \mathcal{E}/\mathcal{E}^3$, $|B| = 3^7$ and $|L/L^3| = 3^{20}$. Proposition 3.3 and Corollary 3.4 hold for $B \otimes B$ as well, replacing integer by integer modulo 3 whenever applicable. We observe $|B \otimes B| \leq 3^{20}$.

Now $\bar{g}, \bar{h}, \bar{g}', \bar{h}' \in B$ have representations analogous to (3.6.3), where all exponents are taken modulo 3. We define now $\psi : B \times B \rightarrow L/L^3$ as $\psi(\bar{g}, \bar{h}) = \psi(g\mathcal{E}^3, h\mathcal{E}^3) = \Phi(g, h) \bmod 3$, where $\Phi(g, h)$ is defined as in (3.6.4). The fact that $\Phi : \mathcal{E} \times \mathcal{E} \rightarrow L$ is well-defined implies that $\psi : B \times B \rightarrow L/L^3$ is well-defined. Since Φ is a crossed pairing, a similar argument shows that ψ is a crossed pairing. Thus by Proposition 1.2 and the fact that ψ is onto, L/L^3 is a homomorphic image of $B \otimes B$. Thus $3^{20} \geq |B \otimes B| \geq |L/L^3| = 3^{20}$, proving our claim. \square

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