

On Commutators (in Groups)

Robert Fitzgerald Morse

University of Evansville
Evansville, Indiana, USA

`rfmorse@evansville.edu`

Joint work with
Luise-Charlotte Kappe (SUNY Binghamton)

Motivation

Let G be a group and let $K(G) = \{[x, y] \mid x, y \in G\}$. The commutator subgroup $G' = \langle K(G) \rangle$.

Theorem 1 (KM). *Let p be a prime and G a group of order p^n . Then $G' = K(G)$, if $n \leq 5$ for odd p and $n \leq 6$ for $p = 2$. Moreover, these bounds are sharp.*

Context

Searching the small groups library in GAP we find that there are two nonisomorphic groups of order 96 such that $K(G) \neq G'$.

All groups of smaller order do not have this property.

A nice project is to prove that all groups of order less than 96 have the property $K(G) = G'$. The p -groups proved to be a challenge.

Our plan was to then to publish a proof that the two groups of order 96 are minimal.

Theorem 2 (Guralnick, 1977). *If (i) G' is abelian and $|G| < 128$ or $|G'| < 16$ or (ii) G' is nonabelian and $|G| < 96$ or $|G'| < 24$, then $K(G) = G'$.*

Context (Continued)

In fact it turns out that many facts about when $K(G)$ is equal or unequal to G' have been “rediscovered” several times in the literature.

We decided to write an expository paper about commutators with a focus on the origins of commutators and look at the literature at various aspects of question when is $K(G)$ equal or unequal to G' .

History

In a letter written by Frobenius in 1896, he writes:

According to Dedekind, I am calling the element F , which is obtained from A and B with the help of the equation $BA = ABF$, the commutator of A and B .

Dedekind was interested in extending character theory from abelian to nonabelian groups. While Dedekind didn't publish any results on the commutator subgroup but he knew a number facts about them such as the commutator subgroup is a normal subgroup and that it is trivial if and only if the group is abelian. It was G. A. Miller who actually published these results. G. A. Miller also credits Dedekind as naming commutators.

History (Continued)

The question as to whether $K(G) = G'$ or not was lurking in the background in the late 19th century.

The first statement of the question whether it is always true that $K(G) = G'$ is in Weber's 1899 textbook. He states that the set of commutators need not equal the commutator subgroup (but does not provide an example).

The first example in the literature of a group G such that $K(G) \neq G'$ is found in a paper of Fite in 1902. Fite attributes the base example to G. A. Miller. He then shows that a homomorphic image of this group also has the property $K(G) \neq G'$.

History (Continued)

Modern notation for commutators is introduced by Levi and van der Waerden in their seminal work on the Burnside groups of exponent 3 in 1933.

Zassenhaus in his 1937 textbook uses the modern notation for commutators and gives many of the basic commutator identities. He states his source is a 1934 paper by P. Hall.

So we have the beginnings of a commutator calculus.

When is $K(G) = G'$

Theorem 3 (Speigel). *Suppose G contains a normal abelian subgroup A with cyclic factor group G/A . Then $K(G) = G'$.*

Theorem 4 (Guralnick). *If G' is an abelian p -subgroup of G with $p > 3$ and $d(G') \leq 3$, then $G' = K(G)$.*

Theorem 5 (Rodney). *If G is nilpotent and G' is cyclic, then $G' = K(G)$.*

Theorem 6 (KM). *Let G be a finite p -group with G' elementary abelian of rank less than or equal to three. Then $K(G) = G'$.*

When is $K(G) \neq G'$

Theorem 7 (Macdonald). *If G is any group and if $|G : Z(G)|^2 < |G'|$, then there are elements in G' that are not commutators.*

Proposition 8 (KM). *Let $p \geq 5$ be a prime and $H = \langle a, b \rangle$ be a nilpotent group of class exactly 4 with $[b, a, b] \in Z(H)$ and $\exp(H') = p$. Then $K(H) \neq H'$.*

Proposition 9 (KM). *Let $H = \langle a, b \rangle$ be a nilpotent group of class exactly 4 with $a^3, b^9, [b, a, b] \in Z(H)$. Then $K(H) \neq H'$.*

Proposition 10 (KM). *Let $H = \langle a, b, c \rangle$ be a group of class 3 precisely. If $a^4, b^2, c^2, [a, c], [b, c]$ and $(ab)^2 \in Z(H)$, then $K(H) \neq H'$.*

Minimal Examples

Types of minimal examples in which $K(G) \neq G'$:

- (i) The smallest group such that G' is abelian (cyclic).
- (ii) The group such that G' is abelian (cyclic) of smallest order.
- (iii) The smallest group such that G' is nonabelian.
- (iv) The group such that G' is nonabelian of order 24.
- (v) The smallest group such that $G' \cap Z(G)$ is generated by noncommutators.
- (vi) The smallest group such that G is perfect.

Any group G of order 128 such that $K(G) \neq G'$ is an example of type (i) and type (ii) where $|G| = 16$. (Of course G' cannot be cyclic here.)

Minimal Examples (Continued)

The smallest group G with cyclic commutator subgroup has order 240 and G' has order 60.

Smallest group with G' is nonabelian is 96 but G' has order 32.

The smallest group G with the smallest nonabelian commutator subgroup has order 216.

Caranti and Scopolla give an example of type (iv) groups of order p^{14} . The smallest such group has order 216.

The smallest perfect group G for which $K(G) \neq G'$ has order 960.

Commutator Length

The function $\lambda(G)$ on a group G is defined as the smallest positive integer n such that every element of G' is a product of n commutators.

If $\lambda(G) > 1$ then of course $K(G) \neq G'$.

Rosenlicht (1962) and Guralnick (1979) find finite upper bounds on $\lambda(G)$ to prove a classical result of Schur that if $[G : Z(G)]$ is finite then G' is finite.

When $[G : Z(G)]$ is finite it follows that G' is finite exactly when $\lambda(G)$ is finite.

Guralnick shows when $[G : Z(G)] = n$ that $\lambda(G) < 3\rho(n)/2$ where $\rho(n)$ is the number of prime divisors of n , counting multiplicity.

Commutator Length (Continued)

Theorem 11 (Macdonald). *For any positive integer n there is a group G such that G' is cyclic and $\lambda(G) > n$.*

There need not be an upper bound for $\lambda(G)$ even for nilpotent groups of class 2.

Cassidy (1979) constructs an infinitely generated nilpotent group G of class 2 group such that for every n there is a element of G' that is a product of n commutators.

Other Topics

Groups with cyclic commutator subgroups. The complexity of this case gives us insight into the complexity into the general cases.

Higher Commutators. When does the r th term of the lower central series consist only of r -fold simple commutators?

Ore's Conjecture

Oystein Ore in 1951 proved for A_n , $n \geq 5$ every element is a commutator. He states the following conjecture which now bears his name:

It is possible that a similar theorem holds for any simple group of finite order, but it seems that at present we do not have the necessary methods to investigate the question.

This result that every element of A_n , $n \geq 5$ is a commutator was proved by G. A. Miller 50 years before and another proof was published the same year (1951) by Ito and a decade later in another paper.

Status of Ore's Conjecture

Name	Verified cases	References	Open cases
$A_n(q), n \geq 1$	all cases	[Tho61], [Tho62a], [Tho62b]	none
$B_2(q)$	all cases	[EG98], [AH85]	none
$B_n(q), n \geq 3$	$q \geq 7$	[EG98]	$q = 2, 3, 4, 5$
$C_n(q), n > 2$	$q \geq 4$	[EG98]	$q = 2, 3$
$D_{2n}(q), n \geq 2$	$q \geq 5$	[EG98]	$q = 2, 3, 4$
$D_{2n+1}(q), n \geq 2$	$q \geq 4$	[EG98]	$q = 2, 3$
$G_2(q)$	all cases	[Bon93]	none
$F_4(q)$	all cases	[Bon93]	none
$E_6(q)$	$q \geq 7$	[EG98]	$q = 2, 3, 4, 5$
$E_7(q)$	$q \geq 5$	[EG98]	$q = 2, 3, 4$
$E_8(q)$	$q \geq 7$	[EG98]	$q = 2, 3, 4, 5$
${}^2A_{2l-1}(q), l > 1$	$q \geq 8$	[EG98]	$q = 2, 3, 4, 5, 7$
${}^2A_{2l}(q), l \geq 1$	$q \geq 4$	[EG98]	$q = 2, 3$
${}^2B_2(q), q = 2^{2m+1}$	all cases	[TL65]	none
${}^2D_n(q), n > 3$	$q \geq 7$	[EG98]	$q = 2, 3, 4, 5$
${}^3D_4(q)$	all cases	[Bon93]	none
${}^2G_2(q)$	all cases	[Bon93]	none
${}^2F_4(q)$	all cases	[Bon93]	none
${}^2E_6(q)$	$q \geq 8$	[EG98]	$q = 2, 3, 4, 5, 7$

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