

Characterizing the n -Engel Property as a Levi-Property

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Definition and Examples

Let \mathfrak{X} be a class of groups. The derived class of groups $L(\mathfrak{X})$ is the class of those groups in which the normal closure of each element in the group is an \mathfrak{X} -group.

The property of being in $L(\mathfrak{X})$ is called the **Levi-property generated by \mathfrak{X}** .

Example. The Dedekind property is the Levi-property generated by the class of cyclic groups.

A group G is a Dedekind group if every subgroup of G is normal

A group G is a Dedekind group if and only if for all $g \in G$, g^G is cyclic.

Definition and Examples (cont)

Proposition 1 (M1999). *The Fitting property is a Levi-property generated by \mathfrak{N} , the class of all nilpotent groups. Symbolically, if \mathfrak{F} is the class of Fitting groups, then $\mathfrak{F} = L(\mathfrak{N})$.*

Proof. Let G be a group in $L(\mathfrak{N})$. Then each element g of G is contained in a normal nilpotent subgroup namely $\langle g \rangle^G$. A group is a Fitting group if and only if every element is contained in a normal nilpotent subgroup. Hence G is a Fitting group.

Suppose G is a Fitting group then each element of G is contained in a normal nilpotent subgroup which contains the normal closure of the element. It follows the normal closure of each element is nilpotent and G is in $L(\mathfrak{N})$. □

In the last two examples involved generating classes that are classes of groups (not just varieties of groups or formations of groups).

Comments

Saying a property is a Levi-property is a weak statement. By saying a group G is in $L(\mathfrak{X})$ we do not gain much structural information about the group. The worse case is the “radical” classes. For example it is not hard to show that for the property of being locally nilpotent we have

$$\text{LN} = L(\text{LN}).$$

We can also look at Levi-properties as a covering property by a system of normal subgroups with a common property. We say that a group is covered by a system of subgroups if it is the set theoretic union of these subgroups.

M. Brodie, L.-C. Kappe, Morse have investigated the consequences of finite coverings.

Varieties of Groups

Let \mathfrak{V} be a variety of groups.

Proposition 2 (M1994). *The class $L(\mathfrak{V})$ is a variety of groups.*

Not all varieties are Levi-varieties.

Proposition 3 (M1994). *Every proper subvariety of zero exponent of the 2-Engel groups i.e. groups that satisfy the law $[x, y, y] = 1$ is not a Levi-variety.*

On the other hand, if the laws of \mathfrak{V} are nice (say recursively defined) then we can describe the laws of $L(\mathfrak{V})$.

Theorem 4 (M1994). *If \mathfrak{V} can be described by outer commutator laws, then the laws of the variety $L(\mathfrak{V})$ can be exactly described.*

Solvable and Nilpotent Varieties

\mathfrak{S}_d is the variety of solvable groups of derived length d .

\mathfrak{N}_c is the variety of nilpotent groups of class c .

Both varieties are defined by outer commutator laws.

Corollary 5. *The variety $L(\mathfrak{N}_c)$ is exactly described by the law*

$$[x^{y_1}, x^{y_2}, \dots, x^{y_c}, x] = 1.$$

Note. Both \mathfrak{N}_c and $L(\mathfrak{N}_c)$ are defined by $c + 1$ variable laws.

n-Engel Variety

\mathfrak{E}_n is the variety of n -Engel groups. These groups satisfy the law

$$[x, \underbrace{y, \dots, y}_n] = [x, {}_n y] = 1.$$

Note. This is not an outer commutator law.

Open Question 6. *Are n -Engel groups locally nilpotent for $n \geq 5$?*

For $n = 2, 3,$ and 4 all n -Engel groups are locally nilpotent. The $n = 4$ case was just recently announced by G. Havas and M. Vaughan-Lee. There are no examples in the literature of an n -Engel group which is not locally nilpotent.

Suppose that there exists a class of groups \mathfrak{X}_n such that $\mathfrak{E}_n = L(\mathfrak{X}_n)$ and \mathfrak{X}_n is locally nilpotent. Then we have that \mathfrak{E}_n is locally nilpotent.

n -Engel as a Levi-variety

The goal to characterize the n -Engel property as a Levi-property generated by \mathfrak{X}_n is in part motivated because it is an approach to answering whether n -Engel groups are locally nilpotent.

How far can we go if \mathfrak{X}_n is a variety? To start there is a connection between $L(\mathfrak{N}_n)$ and \mathfrak{E}_{n+1} :

Each statement below is a consequence of the statement preceding it. For any group G :

- (i) a^G is nilpotent of class at most n for all a in G i.e. $G \in L(\mathfrak{N}_n)$;
- (ii) a^G is n -Engel for all a in G i.e. $G \in L(\mathfrak{E}_n)$;
- (iii) G is $(n + 1)$ -Engel.

The varieties \mathfrak{N}_n and \mathfrak{E}_n make good candidates as generating varieties for \mathfrak{E}_{n+1} . We have the following inclusions

$$L(\mathfrak{N}_n) \leq L(\mathfrak{E}_n) \leq \mathfrak{E}_{n+1}.$$

2-, 3-Engel as a Levi-variety

F. W. Levi (1942) observed that groups in which conjugates commute satisfy the law $[x, y, y] = 1$. As a law

$$[a^g, a^h] = 1 \quad \text{and} \quad [a^g, a] = 1$$

are the same. Now if conjugates commute then the normal closure of each element is abelian. Hence we have $\mathfrak{E}_2 = L(\mathfrak{A})$ where \mathfrak{A} are the class of abelian groups. Therefore, for $n = 1$ (iii) implies (i) above.

The concept of a Levi-property is a generalization of this observation of Levi for 2-Engel groups.

L.-C. Kappe and W. P. Kappe (1972) show that $\mathfrak{E}_3 = L(\mathfrak{N}_2)$.

4-Engel Groups

However for 4-Engel groups we have

$$L(\mathfrak{N}_3) < L(\mathfrak{E}_3) \leq \mathfrak{E}_4.$$

N. Gupta and F. Levin (1980) construct a 4-Engel 5-group G nilpotent of class 6 such that the normal closure for some element in G is nilpotent of class exactly 4. Moreover, the normal closure of each element is 3-Engel.

This example is the best possible in two ways.

Theorem 7 (Traustason (2003), Havas and Vaughan-Lee).

If G is a 4-Engel group, then G is a Fitting group of degree at most 4.

Theorem 8 (M1999). *Let G be an $(n + 1)$ -Engel nilpotent group of class $n + 2$. Then $G \in L(\mathfrak{N}_n)$.*

4-Engel Groups (cont)

One might think well maybe $\mathfrak{E}_4 = L(\mathfrak{N}_4)$.

Let $G = \langle a, b \rangle$ be a free nilpotent of class 5 group. Then the $G \in L(\mathfrak{N}_4)$ but $[a, b, b, b, b] \neq 1$.

Open Question 9. *Is $\mathfrak{E}_4 = L(\mathfrak{E}_3)$.*

This problem is being active worked on by R. Blyth, C. Bussman and myself. All evidence shows that this open problem is true.

Due to a result of Traustason we need only consider 4-Engel groups that have elements of order 5 or 2. And so we are working to play off the torsion restriction to obtain commutator identity we are interesting in.

Solvable n -Engel Groups

In search for sufficient conditions that an n -Engel group is in $L(\mathfrak{N}_n)$, we can look to solvable n -Engel groups. These groups are locally nilpotent.

Theorem 10 (Gruenberg 1953). *A solvable n -Engel group is locally nilpotent.*

The sufficient condition is often an extra commutator law plus the $(n + 1)$ -Engel law which implies the group is in $L(\mathfrak{N}_n)$. This extra commutator law often implies the group is solvable.

A group G is metabelian if G' is abelian. It is solvable of derived length 2. A group G is metabelian if and only if it satisfies the law $[[x_1, x_2], [x_3, x_4]] = 1$. This law just reflects that the generators of G' commute.

Theorem 11 (L.-C. Kappe, M 1990). *If G is a metabelian $(n + 1)$ -Engel group, then $G \in L(\mathfrak{N}_n)$.*

Solvable n-Engel Groups (cont)

Is such a theorem true for solvable groups of derived length 3? The answer is no.

Example 12 (M1999). *Let $k \geq 2$ be an integer and let p be a prime such that $p > k$. Let G be the free nilpotent of class k of exponent p with countable rank. Let $\mathbb{Z}_p G$ be the group ring of G over the integers modulo p and let*

$$M_{p,k} = \left\{ \begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}, \quad g \in G, \quad r \in \mathbb{Z}_p G \right\}.$$

The set $M_{p,k}$ forms a group with the binary operation matrix multiplication with the following properties:

- $M_{p,k}$ is $(p+k)$ -Engel.
- $M_{p,k}$ is solvable of derived length $d = \lfloor \log_2 k \rfloor + 2$.
- $M_{p,k}$ contains an element in which its normal closure is not nilpotent.

Solvable n-Engel Groups (cont)

The group $M_{3,2}$ is an example of a solvable 5-Engel group that is not Fitting and has derived length 3.

However, $M_{p,k}$ is in $L(\mathfrak{E}_{p+k-1})$.

Open Question 13. *If G is a solvable $(n + 1)$ -Engel group must $G \in L(\mathfrak{E}_n)$.*

Other classes of solvable groups

A group G is center-by-metabelian if $G/Z(G)$ is a metabelian group. The variety of center-by-metabelian groups is described by the commutator law $[[x_1, x_2], [x_3, x_4], x_5] = 1$. Such groups are solvable of derived length 3.

Theorem 14 (Brodie, M 2002). *Let G be a $(n + 1)$ -Engel center-by-metabelian group. Then $G \in L(\mathfrak{N}_n)$.*

A group G is called nearly center-by-metabelian if it satisfies the commutator law $[[x_1, x_2, x_3], [x_4, x_5], x_6] = 1$. Such groups are solvable of derived length 4.

Other classes of solvable groups (cont)

Theorem 15 (M1999). *Let G be $(n + 1)$ -Engel nearly center-by-metabelian group. Then $G \in L(\mathfrak{N}_n)$.*

Theorem 16 (M1999). *Let $n \geq 4$ be a natural number and let G be a $(n + 1)$ -Engel group satisfying the commutator law*

$$[[[x_1, x_2, x_3, x_4], [x_5, x_6]], x_7] = 1.$$

Then G is in $L(\mathfrak{N}_n)$.

The bound on n is tight. Since we have a 4-Engel nilpotent group of class 6 which is not in $L(\mathfrak{N}_3)$.

Can we generalize to n -Engel groups which satisfy the commutator identity

$$[[x_1, x_2, \dots, x_k], [y_1, y_2], z] = 1.$$

Is the lower bound on n needed for all n ?

Further Questions

Open Question 17. *For each $n > 3$, does there exist a group G such that the group is n -Engel, nilpotent of class exactly $n + 2$, and G is a Fitting group of degree exactly n ?*

Is it true that if G is $(n + 1)$ -Engel then it is in $L(\mathfrak{E}_n)$?

No. Rips and Shalev (1990) state that there must exist a group G such that G is $(n + 1)$ -Engel and for some x in G its normal closure is not n -Engel. What the bound is on n is not stated. No examples in the literature exist (that I know of) which has this condition.

Open Question 18. *Is the n -Engel property a Levi-variety? Is the n -Engel property a Levi-property?*