

Levi-Properties Generated by Varieties

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ABSTRACT. Levi-properties were first introduced by L. C. Kappe and are modeled after groups investigated by F. W. Levi where conjugates commute. Let \mathfrak{X} be a group theoretic class. A group is in the derived class $L(\mathfrak{X})$ if the normal closure of each element in the group is an \mathfrak{X} -group. The property of being in the class $L(\mathfrak{X})$ is called the Levi-property generated by \mathfrak{X} . In the case where \mathfrak{X} is a variety, we show that $L(\mathfrak{X})$ is also a variety. Given the laws defining any variety \mathfrak{V} , the laws defining a variety \mathfrak{W} can be exactly stated such that $L(\mathfrak{V}) \leq \mathfrak{W}$. However, there exists a variety \mathfrak{V} such that $L(\mathfrak{V}) < \mathfrak{W}$. Our investigations show for varieties defined by outer commutator laws, denoted by \mathfrak{D} , the varieties $L(\mathfrak{D})$ and \mathfrak{W} coincide.

1. Introduction

Given a group theoretic class \mathfrak{X} , we define a derived class of groups $L(\mathfrak{X})$ as the class of those groups in which the normal closure of each element in the group is an \mathfrak{X} -group. The property of being in the class $L(\mathfrak{X})$ is called the Levi-property generated by \mathfrak{X} . In this paper we investigate Levi-properties in which the generating classes are varieties.

Levi-properties were first introduced by L.C. Kappe in [3]. This characterization of groups is modeled after 2-Engel groups first classified by Levi [7]. These groups are exactly those groups in which the normal closure of each element is abelian. Subsequent investigations considered the problem of determining the Levi-property generated by a specific variety of groups such as nilpotency of a given class [1, 4, 6], n -abelian [5], and n -central [6]. In each case, the Levi-properties generated by these varieties are themselves varieties whose laws are derived from the laws of the generating variety. The goal of this paper is to generalize these results.

Fundamental to this investigation is the result that if \mathfrak{V} is a variety of groups, then $L(\mathfrak{V})$ is also a variety (Theorem 2.1). We call $L(\mathfrak{V})$ the Levi-variety generated by \mathfrak{V} . However, not every variety of groups can be presented in the form $L(\mathfrak{V})$ for any \mathfrak{V} (Proposition 2.3).

The variety of all abelian groups, denoted by \mathfrak{A} , is defined by the word $[x, y]$. The law $[x^{y_1}, x^{y_2}] = 1$ holds in the Levi-variety $L(\mathfrak{A})$. We generalize this case and show that, for any variety \mathfrak{V} and for each law which holds in \mathfrak{V} , the same law with each variable in the law replaced with distinct conjugates of a single variable is a law in the Levi-variety generated by \mathfrak{V} (Corollary 3.3). However, the variety defined by these conjugate laws may strictly contain $L(\mathfrak{V})$.

For the abelian case, $L(\mathfrak{A})$ is exactly characterized by the conjugate law $[x^{y_1}, x^{y_2}] = 1$. A similar characterization is given for Levi-varieties generated by varieties defined by the outer commutator words of P. Hall [2] (Corollary 4.4).

2. Levi-varieties

Our notation is standard (see, for instance, [10]). We fix $x, x_1, \dots, y, y_1, \dots$ to be variables and \mathfrak{w} and \mathfrak{v} to be words in these variables. Denote the variety of nilpotent groups of class c by \mathfrak{N}_c . The variety of n -Engel groups, defined by the commutator word $[x, \underbrace{y, \dots, y}_n]$, is denoted by \mathfrak{E}_n .

The following result is fundamental to our investigation.

THEOREM 2.1. *If \mathfrak{V} is a variety of groups, then $L(\mathfrak{V})$ is also a variety.*

PROOF. A straightforward argument shows that the class $L(\mathfrak{V})$ is closed under the formation of subgroups, homomorphic images, and Cartesian products. It follows by a result of Birkhoff that $L(\mathfrak{V})$ is a variety (see [9, Theorem 15.23]). \square

Distinct classes of groups may not generate distinct Levi-properties. The normal closure of each element of a group being an \mathfrak{X} -group may put a constraint on the class \mathfrak{X} as it can occur in the normal closures. This constraint may act on two distinct classes of groups such that both will generate the same Levi-property. For example, \mathfrak{E}_2 strictly contains \mathfrak{N}_2 since there exist 2-Engel groups which are nilpotent of exactly class 3 [7]. However, each of these varieties generate the same Levi-variety, namely \mathfrak{E}_3 [4]. That is, 2-Engel groups are constrained as they appear in the normal closures to those 2-Engel groups which are nilpotent of class less than 3. Hence, the varieties \mathfrak{E}_2 and \mathfrak{N}_2 are equivalent as they occur in the normal closure of each element in a group. We denote by $\overline{\mathfrak{X}}$ those \mathfrak{X} -groups which occur in the normal closures of each group in $L(\mathfrak{X})$. Note that $\overline{\mathfrak{X}} \subseteq \mathfrak{X}$. The lemma below describes relations and implications between classes of groups and the Levi-properties they generate.

LEMMA 2.2. *Let \mathfrak{X} and \mathfrak{Y} be classes of groups. Then we have the following implications:*

- (i) *If \mathfrak{X} is inherited by normal subgroups, then $\mathfrak{X} \leq L(\mathfrak{X})$.*
- (ii) *If $\mathfrak{X} \leq \mathfrak{Y}$, then $L(\mathfrak{X}) \leq L(\mathfrak{Y})$. (But $\mathfrak{X} < \mathfrak{Y}$ does not necessarily imply $L(\mathfrak{X}) < L(\mathfrak{Y})$ as noted above.)*
- (iii) *If $L(\mathfrak{X}) \leq L(\mathfrak{Y})$, then $\overline{\mathfrak{X}} \leq \overline{\mathfrak{Y}}$. In particular, $L(\mathfrak{X}) < L(\mathfrak{Y})$ implies $\overline{\mathfrak{X}} < \overline{\mathfrak{Y}}$.*

PROOF. Let \mathfrak{X} and \mathfrak{Y} be classes of groups.

- (i) Obvious.
- (ii) Let $G \in L(\mathfrak{X})$. Hence, $z^G \in \overline{\mathfrak{X}} \leq \mathfrak{X}$ for all $z \in G$. If $\mathfrak{X} \leq \mathfrak{Y}$, then $z^G \in \mathfrak{Y}$ for all $z \in G$ which implies $z^G \in \overline{\mathfrak{Y}}$ and $G \in L(\mathfrak{Y})$. Thus, $L(\mathfrak{X}) \leq L(\mathfrak{Y})$.
- (iii) Assume $L(\mathfrak{X}) \leq L(\mathfrak{Y})$. Let $G \in L(\mathfrak{X})$. Then $z^G \in \overline{\mathfrak{X}}$ for all $z \in G$. Suppose $z_0^G \notin \overline{\mathfrak{Y}}$ for some $z_0 \in G$. This implies $G \notin L(\mathfrak{Y})$, a contradiction. If $L(\mathfrak{X}) < L(\mathfrak{Y})$, then there exists a group $G \in L(\mathfrak{Y})$ such that there exists a $z \in G$ such that z^G is not an $\overline{\mathfrak{X}}$ -group. Hence, we have strict containment, i.e., $\overline{\mathfrak{X}} < \overline{\mathfrak{Y}}$ \square .

Every variety generates a Levi-variety. However, not all varieties are Levi-varieties.

PROPOSITION 2.3. *Every proper subvariety of zero exponent of \mathfrak{E}_2 is not a Levi-variety.*

PROOF. Let \mathfrak{V} be a proper subvariety of zero exponent of \mathfrak{E}_2 . Suppose that there exists a variety \mathfrak{X} such that $L(\mathfrak{X}) = \mathfrak{V}$. Then \mathfrak{X} must be of zero exponent and hence \mathfrak{X} contains the variety of abelian groups. Therefore, by Lemma 2.2 (ii), we have $\mathfrak{E}_2 = L(\mathfrak{A}) \leq L(\mathfrak{X}) = \mathfrak{V}$. This contradicts the assumption that \mathfrak{V} is a proper subvariety of \mathfrak{E}_2 .

3. Laws defining Levi-varieties

In light of Theorem 2.1, can one specifically state all the laws of $L(\mathfrak{V})$ if those of \mathfrak{V} are given? In general, this is an open question. However, laws can be derived from the laws defining \mathfrak{V} which always hold in $L(\mathfrak{V})$ by replacing each variable in these laws with distinct conjugates of a single variable. Moreover, if \mathfrak{w} is an n -variable law of \mathfrak{V} , then the law derived from \mathfrak{w} is also an n -variable law.

The following lemma shows a certain amount of freedom is allowed when working with laws concerning conjugates of elements.

LEMMA 3.1. *Let G be any group. Then G satisfies the law*

$$\mathfrak{w}(x^{y_1}, \dots, x^{y_{k-1}}, x, x^{y_{k+1}}, \dots, x^{y_{n+1}}) = 1$$

if and only if G satisfies the law $\mathfrak{w}(x^{y_1}, \dots, x^{y_k}, \dots, x^{y_{n+1}}) = 1$.

THEOREM 3.2. *Let $\mathfrak{w}(x_1, \dots, x_n)$ be a word and \mathfrak{V} be the variety defined by \mathfrak{w} . Then the n -variable word $\mathfrak{w}(x^{y_1}, \dots, x^{y_{n-1}}, x)$ is a law of $L(\mathfrak{V})$.*

PROOF. Let \mathfrak{V} be the variety defined by the word $\mathfrak{w}(x_1, \dots, x_n)$. Suppose $G \in L(\mathfrak{V})$. Then for all $z \in G$, $z^G \in \mathfrak{V}$, \mathfrak{w} reduces to the identity element for $z^{g_1}, z^{g_2}, \dots \in z^G$, where g_1, g_2, \dots are arbitrary elements of G . Hence, $\mathfrak{w}(x^{y_1}, \dots, x^{y_n})$ is a law in G . By Lemma 3.1, this law is equivalent to the n -variable law

$$\mathfrak{w}(x^{y_1}, \dots, x^{y_{n-1}}, x) = 1. \quad \square$$

The following corollary is now obvious.

COROLLARY 3.3. *Let Λ be an index set and $W = \{\mathfrak{w}_\lambda | \lambda \in \Lambda\}$ be a set of words in variables $\{x_1, x_2, \dots\}$. If \mathfrak{V} is the variety defined by the set of words W , then $L(\mathfrak{V})$ is a subvariety of the variety \mathfrak{W} defined by the set of words W , where the variables $\{x_1, x_2, \dots\}$ are replaced by $\{x^{y_1}, x^{y_2}, \dots\}$.*

We conclude this section with an example for which $L(\mathfrak{V})$ is a strict subvariety of \mathfrak{W} .

EXAMPLE 3.4. Let $W = \{[x_1^2, x_2]\}$ and \mathfrak{V} be the variety defined by W . Then \mathfrak{V} is the variety of 2-central groups, and \mathfrak{W} is defined by the law $[x^2, x^y]$. Let G be the infinite dihedral group $G = \langle h, i : i^2 = 1, h^i = h^{-1} \rangle$. We first show that $G \in \mathfrak{W}$. If $x \in G$, then either $x \in \langle h \rangle$, so the normal closure of x is abelian, or $x = h^n i$ for some integer n , in which case $x^2 = 1$. In both cases, $[x^2, x^y] = 1$ for all $y \in G$ as required. However, G is not a 3-Engel group. Therefore, $G \notin L(\mathfrak{W})$ [6, Theorem 8].

4. Outer Commutator Laws

Let $L(\mathfrak{V})$ and \mathfrak{W} be varieties as defined in Corollary 3.3. It is not always the case that $L(\mathfrak{V}) < \mathfrak{W}$. The varieties $L(\mathfrak{V})$ and \mathfrak{W} may coincide when the generating variety \mathfrak{V} is defined by certain words. For example, the two varieties coincide when the generating variety is defined by single variable words, i.e., $\mathfrak{w}(x) = x^n$. This is an immediate consequence of Lemma 3.1. In this section, we show that this is also true for generating varieties defined by outer commutator words. Our treatment of outer commutator words follows Robinson [10].

We define the weight of a word as the number of distinct variables it contains. The unique outer commutator word of weight one is Θ_0 where $\Theta_0(x_1) = x_1$. The word Θ is an outer commutator word of weight $n > 1$ if $\Theta = [\Theta_1, \Theta_2]$, where Θ_1 and Θ_2 are outer commutator words of weight n_1 and n_2 respectively and $n = n_1 + n_2$. Since each variable of an outer commutator word appears at most once, we see the lower central words are outer commutator words whereas the n -Engel words ($n > 1$) are not.

Let G be a group and $z \in G$. Every element $u \in z^G$ can be expressed as a finite product of $z^{\pm g_i}$, $g_i \in G$. Using the length of this product, we define the function $\eta : z^G \rightarrow \{\mathbb{N} \cup 0\}$ as follows:

$$\eta(u) = \begin{cases} 0, & \text{for } u = 1 \\ k, & \text{for } u = z^{\pm g_1} \dots z^{\pm g_k}, g_i \in G, \text{ and } k \text{ minimal} \end{cases}$$

We observe that the function η is not affected by conjugation:

$$(1) \quad \eta(u^g) = \eta(u) \quad u \in z^G, g \in G.$$

Let $z \in G$ and $u_i \in z^G$. We define the total length \mathcal{L} of a word \mathfrak{w} with entries u_1, \dots, u_j as

$$\mathcal{L}(\mathfrak{w}(u_1, \dots, u_j)) = \sum_{i=1}^j \eta(u_i).$$

Two words with equal weight and identical entries from z^G have equal total length.

For any word $\mathfrak{w}(x_1, \dots, x_j)$, we have

$$(2) \quad \mathfrak{w}(z_1, \dots, z_j)^h = \mathfrak{w}(z_1^h, \dots, z_j^h) \quad z_i, h \in G.$$

With this observation, it follows from (1) that the total length of a word \mathfrak{w} is not affected by conjugation, that is, for $z, g \in G$ and $u_i \in z^G$,

$$(3) \quad \mathcal{L}(\mathfrak{w}(u_1, \dots, u_j)^g) = \mathcal{L}(\mathfrak{w}(u_1^g, \dots, u_j^g)) = \mathcal{L}(\mathfrak{w}(u_1, \dots, u_j)).$$

The following two lemmas will facilitate the proof of our main result.

LEMMA 4.1. *Let Θ be an outer commutator word of weight n , and let G be a group.*

- (i) *Let $z, g_i \in G$, $i = 1, \dots, n$. Then there exist $h_i \in G$, $i = 1, \dots, n$, such that*

$$\Theta(z^{g_1}, \dots, z^{-g_k}, \dots, z^{g_n}) = \Theta(z^{h_1}, \dots, z^{h_k}, \dots, z^{h_n})^{-1}.$$

- (ii) *If G satisfies the law $\Theta(x^{y_1}, \dots, x^{y_n}) = 1$, then G also satisfies the law*

$$\Theta(x^{\delta_1 y_1}, \dots, x^{\delta_n y_n}) = 1$$

where $\delta_i = \pm 1$, $i = 1, \dots, n$.

PROOF.

- (i) We prove this statement by induction on n , the weight of Θ . Let G be a group, and let $z, g_i \in G$, $i = 1, \dots, n$. For $n = 1$, we have $\Theta_0(z^{-g_1}) = z^{-g_1}$. Relabeling g_1 as h_1 , we see (i) is true for $n = 1$.

Assume (i) is true for all outer commutator words of weight less than n . Suppose Θ is an outer commutator of weight n . Then by definition, we have outer commutators Φ and Γ of weight $i > 0$ and $n - i$ respectively such that $\Theta = [\Phi, \Gamma]$.

Without loss of generality, assume $1 \leq k \leq i$. We apply the induction hypothesis to the outer commutator Φ , and hence, there exist $\bar{h}_j \in G$, $j = 1, \dots, i$, such that

$$\Phi(z^{g_1}, \dots, z^{-g_k}, \dots, z^{g_i}) = \Phi(z^{\bar{h}_1}, \dots, z^{\bar{h}_i})^{-1}.$$

Hence,

$$\Theta(z^{g_1}, \dots, z^{-g_k}, \dots, z^{g_n}) = \left[\Phi(z^{\bar{h}_1}, \dots, z^{\bar{h}_k}, \dots, z^{\bar{h}_i})^{-1}, \Gamma(z^{g_{i+1}}, \dots, z^{g_n}) \right].$$

By standard commutator expansion and relabeling, we obtain the desired result. A similar argument holds if $i + 1 \leq k \leq n$.

- (ii) Suppose the law $\Theta(x^{y_1}, \dots, x^{y_n}) = 1$ holds in G . For $z, g_i \in G$ and $\delta_i = \pm 1$, $i = 1, \dots, n$, we obtain

$$\Theta(z^{\delta_1 g_1}, \dots, z^{\delta_n g_n}) = \Theta(z^{h_1}, \dots, z^{h_n})^{\pm 1}$$

by applying (i) to each $\delta_i = -1$, $i = 1, \dots, n$. By hypothesis, we have

$$\Theta(z^{h_1}, \dots, z^{h_n})^{\pm 1} = 1.$$

Therefore, G satisfies the desired law. \square

LEMMA 4.2. *Let G be a group, $z \in G$ and Θ be an outer commutator word of weight n . Let $1 \neq u_i \in z^G$, $i = 1, \dots, n$, and $\eta(u_k) \geq 2$ for some $k \in \{1, \dots, n\}$. Then there exist $v_i, w_i \in z^G$, $i = 1, \dots, n$, with*

$$\Theta(u_1, \dots, u_n) = \Theta(v_1, \dots, v_n) \Theta(w_1, \dots, w_n)$$

and

$$\begin{aligned} 0 &< \mathcal{L}(\Theta(v_1, \dots, v_n)) < \mathcal{L}(\Theta(u_1, \dots, u_n)), \\ 0 &< \mathcal{L}(\Theta(w_1, \dots, w_n)) < \mathcal{L}(\Theta(u_1, \dots, u_n)). \end{aligned}$$

PROOF. We prove the lemma by induction on n , the weight of Θ . Let G be a group and $z \in G$. A straightforward argument shows that the lemma is true for $n = 1$.

Assume the result is true for all outer commutator words of weight less than n . Suppose Θ is an outer commutator of weight n . Then by definition, we have outer commutators Φ and Γ of weight less than n such that for variables x_1, \dots, x_n

$$\Theta(x_1, \dots, x_n) = [\Phi(x_1, \dots, x_i), \Gamma(x_{i+1}, \dots, x_n)].$$

Let $1 \neq u_j \in z^G$ for $j = 1, 2, \dots, n$, and assume $\eta(u_k) \geq 2$ for some $1 \leq k \leq i$. We apply the induction hypothesis to the outer commutator Φ , and hence, there exist $\bar{v}_j, w_j \in z^G$, $j = 1, \dots, i$, such that

$$\Phi(u_1, \dots, u_i) = \Phi(\bar{v}_1, \dots, \bar{v}_i) \Phi(w_1, \dots, w_i)$$

and

$$(4) \quad 0 < \mathcal{L}(\Phi(\bar{v}_1, \dots, \bar{v}_i)) < \mathcal{L}(\Phi(u_1, \dots, u_i)),$$

$$(5) \quad 0 < \mathcal{L}(\Phi(w_1, \dots, w_i)) < \mathcal{L}(\Phi(u_1, \dots, u_i)).$$

Set $\Phi(w_1, \dots, w_i) = c$ whenever an element is conjugated by $\Phi(w_1, \dots, w_i)$. By standard commutator expansion, we obtain

$$\begin{aligned} \Theta(u_1, \dots, u_n) &= [\Phi(\bar{v}_1, \dots, \bar{v}_i) \cdot \Phi(w_1, \dots, w_i), \Gamma(u_{i+1}, \dots, u_n)] \\ &= [\Phi(\bar{v}_1, \dots, \bar{v}_i), \Gamma(u_{i+1}, \dots, u_n)]^c [\Phi(w_1, \dots, w_i), \Gamma(u_{i+1}, \dots, u_n)]. \end{aligned}$$

By (2) we see

$$(6) \quad \Theta(u_1, \dots, u_n) = [\Phi(\bar{v}_1^c, \dots, \bar{v}_i^c), \Gamma(u_{i+1}^c, \dots, u_n^c)] [\Phi(w_1, \dots, w_i), \Gamma(u_{i+1}, \dots, u_n)].$$

By (3) we have

$$0 < \mathcal{L}([\Phi(\bar{v}_1^c, \dots, \bar{v}_i^c), \Gamma(u_{i+1}^c, \dots, u_n^c)]) = \mathcal{L}([\Phi(\bar{v}_1, \dots, \bar{v}_i), \Gamma(u_{i+1}, \dots, u_n)]),$$

and hence, by (4)

$$0 < \mathcal{L}([\Phi(\bar{v}_1^c, \dots, \bar{v}_i^c), \Gamma(u_{i+1}^c, \dots, u_n^c)]) < \mathcal{L}([\Phi(u_1, \dots, u_i), \Gamma(u_{i+1}, \dots, u_n)]).$$

It follows by (5) that

$$0 < \mathcal{L}([\Phi(w_1, \dots, w_i), \Gamma(u_{i+1}, \dots, u_n)]) < \mathcal{L}([\Phi(u_1, \dots, u_i), \Gamma(u_{i+1}, \dots, u_n)]).$$

The desired decomposition is obtained by relabeling the group elements of $\Theta(u_1, \dots, u_n)$. Setting $\bar{v}_j^c = v_j$ and $u_j = w_j$ for $j = 1, \dots, i$, and $u_j^c = w_j$ for $j = i + 1, \dots, n$, in (6) proves the lemma for $1 \leq k \leq i$. A similar argument holds if $i + 1 \leq k \leq n$. \square

From the following theorem, we see that if \mathfrak{D} is a variety defined by an n -variable outer commutator word, then $L(\mathfrak{D})$ is a variety defined by an n -variable word.

THEOREM 4.3. *Let $\Theta(x_1, \dots, x_n)$ be an outer commutator word and \mathfrak{D} be the variety defined by Θ . Then $L(\mathfrak{D})$ is the variety defined by the n -variable word $\Theta(x^{y_1}, \dots, x^{y_{n-1}}, x)$.*

PROOF. Let G be a group. Suppose $G \in L(\mathfrak{D})$. Then, by Theorem 3.2, the law $\Theta(x^{y_1}, \dots, x^{y_{n-1}}, x) = 1$ holds in G .

Now suppose the group G is in the variety defined by the word

$$\Theta(x^{y_1}, \dots, x^{y_{n-1}}, x).$$

It suffices to show that, for every $z \in G$, the law $\Theta(x_1, \dots, x_n) = 1$ holds in z^G . It follows that $z^G \in \mathfrak{D}$ and consequently $G \in L(\mathfrak{D})$.

By induction on the total length of Θ , we will show $\Theta(u_1, \dots, u_n) = 1$ for $z \in G$ and every $u_1, \dots, u_n \in z^G$. If $\eta(u_i) = 0$ for some $i \in \{1, \dots, n\}$, then u_i is the identity element. It follows that $\Theta(u_1, \dots, u_n) = 1$ since Θ is a commutator word. Suppose $\mathcal{L}(\Theta(u_1, \dots, u_n)) < n$. Then $\eta(u_i) = 0$ for at least one $i \in \{1, \dots, n\}$ and hence $\Theta(u_1, \dots, u_n) = 1$. From now on, we assume, without loss of generality, that $\eta(u_i) > 0$ for $i = 1, \dots, n$. Suppose $\mathcal{L}(\Theta(u_1, \dots, u_n)) = n$. Then $\eta(u_i) = 1$ for each $i = 1, \dots, n$. Hence, it follows that $u_i = z^{\pm g_i}$. Since G satisfies the law $\Theta(x^{y_1}, \dots, x^{y_n}) = 1$ by hypothesis, we can apply Lemma 4.1(ii) and thus $\Theta(u_1, \dots, u_n) = 1$.

Suppose the claim is true when the total length of Θ is less than m , $m > n$. We now show the result for $\mathcal{L}(\Theta(u_1, \dots, u_n)) = m$. Since m is larger than n , there is at least one u_i , $i \in \{1, \dots, n\}$, such that $\eta(u_i) \geq 2$. By Lemma 4.2 there exist $v_i, w_i \in z^G$, $i = 1, \dots, n$, with $\Theta(u_1, \dots, u_n) = \Theta(v_1, \dots, v_n)\Theta(w_1, \dots, w_n)$ such that

$$\begin{aligned} 0 < \mathcal{L}(\Theta(v_1, \dots, v_n)) < \mathcal{L}(\Theta(u_1, \dots, u_n)) = m, \\ 0 < \mathcal{L}(\Theta(w_1, \dots, w_n)) < \mathcal{L}(\Theta(u_1, \dots, u_n)) = m. \end{aligned}$$

By the induction hypothesis, $\Theta(v_1, \dots, v_n) = 1$ and $\Theta(w_1, \dots, w_n) = 1$. Hence $\Theta(u_1, \dots, u_n) = 1$. Therefore, for any $u_1, u_2, \dots, u_n \in z^G$, we have $\Theta(u_1, \dots, u_n) = 1$ as desired. \square

COROLLARY 4.4. *Let Λ be an index set and $W = \{\Theta_\lambda | \lambda \in \Lambda\}$ be a set of outer commutator words in variables $\{x_1, x_2, \dots\}$. If \mathfrak{D} is the variety defined by the set of words W , then $L(\mathfrak{D})$ is the variety defined by the set of words W in variables $\{x^{y_1}, x^{y_2}, \dots\}$.*

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