The third homotopy group of the suspension of an Eilenberg-Mac Lane space for Coxeter and spherical Artin groups

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Abstract

We compute the homotopy group $\pi_3(SK(W,1))$ for all finite reflection groups W using a method that can be applied to any Coxeter group. We also calculate $\pi_3(SK(\tilde{W},1))$ for all Artin groups $\tilde{W}$ of finite type.

1 Introduction

Let $SK(G,1)$ be the space formed by taking the suspension of an Eilenberg-Mac Lane space of a group G. It is a standard fact that the suspension is simply connected and has second homotopy group $\pi_2 SK(G,1) = G_{ab}$. In this paper we prove the following result on $\pi_3 SK(G,1)$.

Theorem 1 *For any Coxeter group W we have*

$$\pi_3(SK(W,1)) \cong (\mathbb{Z}_2)^l \oplus (\mathbb{Z}_4)^m$$

where l, m are integers ≥ 0. If W is finite then its associated Artin group $\tilde{W}$ is such that

$$\pi_3(SK(\tilde{W},1)) \cong \mathbb{Z}^p \oplus (\mathbb{Z}_2)^q \oplus (\mathbb{Z}_4)^r$$

where p, q, r are integers ≥ 0. For finite W we have $m = 0$ and the integers l, p, q, r are given in the following table.

Type =	$A_n (n \geq 1)$	$B_n (n \geq 2)$	$D_n (n \geq 4)$	E_6	E_7	E_8	F_4	H_3	H_4	$I_2(m)$
$l =$	2									
$p =$										
$q =$										
$r =$										

Our main tool is J.H.C. Whitehead's [**?**] exact sequence

$$\ldots \to \pi_4 X \to H_4(X) \to \Gamma(\pi_2 X) \to \pi_3 X \to H_3(X) \to 0 \tag{1}$$

relating the homotopy groups and integral homology groups of a simply connected space X. For any abelian group A the group $\Gamma(A)$ is generated by symbols $\gamma(a)$ $(a \in A)$ subject to relations asserting that the expression $\gamma(a+b)-\gamma(a)-\gamma(b)$ is bilinear in $a,b\in A$. On taking $X=SK(G,1)$ sequence (1) yields the exact sequence

$$H_3(G)\xrightarrow{\beta}\Gamma(G_{ab})\to\pi_3SK(G,1)\xrightarrow{\alpha}H_2(G)\to 0\ . \tag{2}$$

Howlett [?] has calculated $H_2(W)$ for an arbitrary Coxeter group W and, in particular, shown it to be an elementary abelian 2-group. By definition (see Section 3) W is finitely generated by elements of order 2. It thus follows from sequence (2) and Lemma 2 below that $\pi_3(SK(W,1))\cong(\mathbb{Z}_2)^l\oplus(\mathbb{Z}_4)^m$ for some $l,m\geq 0$. Using an explicit free $\mathbb{Z}W$-resolution of the integers we determine the cokernel of β and solve an extension problem to obtain the values of l,m for finite W. The method can be applied to arbitrary W.

The method can also be applied to certain Artin groups $\tilde{W}$. It is known that $H_2(\tilde{W})$ is finitely generated with finite part an elementary abelian 2-group when $\tilde{W}$ is of finite type (see for example [?]). Using an explicit free $\mathbb{Z}\tilde{W}$-resolution with sequence (2) we determine $H_2(\tilde{W})$ in this case.

The homotopy group $\pi_3(SK(G,1))$ is related via an exact sequence

$$0\to\pi_3(SK(G,1))\to G\otimes G\to G\to G_{ab}\to 0$$

to the nonabelian tensor square of Brown and Loday [?]. The structure of $G\otimes G$ can be obtained from that of $\pi_3(SK(G,1))$ and the commutator subgroup $[G,G]$ by solving an extension problem. We end the paper with a couple of examples illustrating this new approach to calculating the tensor square.

2 Nonabelian tensor products

We shall express some of our calculations using the language of nonabelian tensor products. Recall [?] that the *tensor product* $N\otimes G$ of a group G with a normal subgroup N is a group generated by symbols $n\otimes g$ ($n\in N, g\in G$) subject to relations

$$nn'\otimes g={}^{n}(n'\otimes g)\ (n\otimes g)$$

$$n\otimes gg'=(n\otimes g)\ {}^{g}(n\otimes g')$$

for $n,n'\in N, g,g'\in G$. Here ${}^{g}(n\otimes g')=gng^{-1}\otimes gg'g^{-1}$. There is a homomorphism $\kappa\colon N\otimes G\to G, n\otimes g\mapsto ngn^{-1}g^{-1}$. One defines $J(G)=\ker(\kappa\colon G\otimes G\to G)$. An isomorphism $J(G)\cong\pi_3SK(G,1)$ was proved in [?]. Sequence (2) can thus be written as a sequence of purely algebraic terms:

$$H_3(G)\xrightarrow{\beta}\Gamma(G_{ab})\to J(G)\xrightarrow{\alpha}H_2(G)\to 0\ . \tag{3}$$

The *exterior product* $N\wedge G$ is a quotient of the tensor product obtained by imposing the relations

$$n\otimes n=1$$

for $n\in N$. There is again a homomorphism $\kappa'\colon N\wedge G\to G$.

An isomorphism $\iota\colon H_3(G)\xrightarrow{\cong}\ker(\kappa'\colon R\wedge F\to F)$ was established in [?] for any free group F with normal subgroup R and quotient $F/R\cong G$. The proof in [?] is topological but a a purely algebraic proof was given in [?]. Moreover, an algebraic proof of the exact sequence

(3) was given in [?]. The algebraic proofs provide an explicit description of the map β as a composite homomorphism

$$\beta\colon H_3(G) \xrightarrow{\iota} \ker(\kappa'\colon R \wedge F \to F) \xrightarrow{\beta'} \Gamma(G_{ab}) \ .$$

For Lemma 2 below we need an explicit description of β'. An element $h \in \ker(\kappa')$ can be written as a product $h = \prod_{i=1}^{k} r_i \wedge f_i$ in $R \wedge F$. This corresonds to a product $\tilde{h} = \prod_{i=1}^{k} r_i \otimes f_i$ in $F \otimes F$. Since $\kappa(\tilde{h}) = 1 \in F$ general theory implies that one can express $\tilde{h}$ in the form $\tilde{h} = \prod_{j=1}^{m} f_j \otimes f_j$. The homomorphism β' is defined by $\beta'(h) = \sum_{j=1}^{m} \gamma(f_j)$.

Suppose now that the group G is generated by a set $\underline{x}$ for which each generator $x \in \underline{x}$ satisfies $x^e = 1$. The abelianized group G_{ab} is generated by the cosets $x[G,G]$ for $x \in \underline{x}$. One readily sees that the group $\Gamma(G_{ab})$ is generated by the elements $\gamma(x[G,G])$ for $x \in \underline{x}$. Take F to be the free group on $\underline{x}$ and let R be the kernel of the canonical quotient $F \to G$. The element $x^e \wedge x$ lies in $\ker(\kappa'\colon R \wedge F \to F)$ and gets mapped by β' to $e\gamma(x)$. This proves the following lemma.

Lemma 2 *Let G be generated by a finite set of generators with orders dividing e. Then the exponent of $\Gamma(G_{ab})/\beta H_3(G)$ divides e.*

3 Coxeter groups, Artin groups and resolutions

A *Coxeter diagram* D is a graph with vertex set V and edge set $E \subset V \times V$. Each edge $e \in E$ is labelled by an integer $\lambda(e) \geq 3$ or by the symbol $\lambda(e) = \infty$. The associated *Coxeter group* W_D is defined by a presentation on generators x_v $(v \in V)$ subject to relations $x_v^2 = 1$ for $v \in V$ and the relations $(x_u x_v)^{\lambda(e)}$ for $e = (u,v) \in E$.

We denote the generating set of W by S and identify it with V. The pair (W,S) is referred to as a *Coxeter system.* In pictures of D we only include those edge labels $\lambda(e) \geq 4$. The label $\lambda(e) = 3$ occurs often and is thus omitted.

Finite Coxeter groups are classified and coincide with the finite reflection groups [?]. A polynomial growth free resolution has been obtained for finite Coxeter groups by de Concini and Salvetti [?]. Their proof uses results of Deligne and Salvetti on the topology of hyperplane arrangements. A direct proof is given in [?] using a group action on a certain convex polytope. The proof in [?] generalizes to yield a free $\mathbb{Z}W$-resolution $F_*^W \to \mathbb{Z}$ for an arbitrary Coxeter group W. In the general proof the action on a polytope is replaced by an action of W on the so-called Davis complex (see [?] for details). The homology of W is by definition the homology of the chain complex $C_*^W = F_*^W \otimes_{\mathbb{Z}W} \mathbb{Z}$. For the proof of Theorem 1 we will need a description of this chain complex C_*^W in dimensions ≤ 3.

Let (W,S) be an arbitrary Coxeter system corresponding to the diagram D. A *k-multiset* on S is an unordered collection of k elements chosen from S with replacement. We say that a k-multiset U on S is of *finite type* if the elements of U generate a finite subgroup of W. The chain group C_k^W is the free abelian group whose free generators correspond to the finite type k-multisets on S. We denote by $[U]$ the free generator corresponding to the multiset U. We can define the boundary homomorphism $d_k\colon C_k^W \to C_{k-1}^W$ by its image on free generators $[U]$. We recall the definition for $0 \leq k \leq 3$.

The formula for $d_k[U]$ is determined by the multiplicity of the elements in U together with the full subgraph D_U of D determined by those elements. Consider for example the finite

Coxeter group W associated to the diagram $D = \bullet - \bullet - \bullet$. In this case $|S| = 3$ and C_3^W has rank 10. Three generators of C_3^W have the form $[x^3\bullet]$ for $x \in S$. Four generators have the form $[x^2 \bullet - \bullet\, y]$ for $x, y \in S$. Two generators have the form $[x^2 \bullet \quad \bullet\, y]$ for $x, y \in S$. One generator has the form $[x \bullet - \bullet\, y - \bullet z]$ for $x, y, z \in S$. Since any generator has one of four forms the homomorphism d_k can be specified by just four formulae.

The following formulae determine d_k $(0 \leq k \leq 3)$ for an arbitrary Coxeter group W. In the two formulae involving a minus sign we assume that $x < y$.

$$\begin{array}{lll}
d_1[x\bullet] = & 0 & \\
d_2[x \bullet -^m \bullet\, y] = & [x\bullet] - [y\bullet] & (m \text{ odd }) \\
d_2[x \bullet -^m \bullet\, y] = & 0 & (m \text{ even }) \\
d_2[x \bullet \quad \bullet\, y] = & 0 & \\
d_2[x^2\bullet] = & 2[x\bullet] & \\
d_3[x \bullet - \bullet\, y - \bullet z] = & 2[x \bullet \quad \bullet\, z] & \\
d_3[x \bullet - \bullet\, y -^4 \bullet z] = & 0 & \\
d_3[x \bullet - \bullet\, y -^5 \bullet z] = & 0 & \\
d_3[x \bullet \quad \bullet\, y \quad \bullet\, z] = & 0 & \\
d_3[x^2 \bullet -^m \bullet\, y] = & [x^2\bullet] - [y^2\bullet] & \\
d_3[x^2 \bullet \quad \bullet\, y] = & 0 & \\
d_3[x^3\bullet] = & 0 &
\end{array}$$

Associated to a Coxeter diagram D is the Artin group $\tilde{W}$ defined by a presentation on generators x_v $(v \in V)$ subject to relations $(x_u x_v)^{\lambda(e)}$ for $e = (u, v) \in E$. The corresponding Coxeter group W is obtained by adding the relations $x_v^2 = 1$ for $v \in V$. If W is finite (and in many other cases too) there is an explicit polynomial growth free $\mathbb{Z}\tilde{W}$-resolution $F_*^{\tilde{W}} \to \mathbb{Z}$ due to Squier, Salvetti and others (see [**?**]). The homology of $\tilde{W}$ is by definition the homology of the chain complex $C_*^{\tilde{W}} = F_*^{\tilde{W}} \otimes_{\mathbb{Z}\tilde{W}} \mathbb{Z}$. The chain group $C_k^{\tilde{W}}$ is the free abelian group whose free generators correspond to the finite type subsets $U \subset S$ of size k. We denote by $[U]$ the free generator corresponding to the subset U. The boundary homomorphism $d_k \colon C_k^{\tilde{W}} \to C_{k-1}^{\tilde{W}}$ is defined by its image on free generators $[U]$ by the appropriate formulae above. Since a subset U has no repeated element the above formulae involving x^2 or x^3 play no part in the boundary map.

4 The homomorphisms α and ι

5 Proof of theorem 1

6 Calculating tensor squares