# Advances in computing the nonabelian tensor square of polycyclic groups

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# Definition

Given a group $G$ we define the nonabelian tensor square $G \otimes G$ as the group generated by the symbols

$$g \otimes g' \quad \text{for all } g, g' \in G$$

subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$$
$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

where ${}^{x}y = xyx^{-1}$.

The nonabelian tensor square is a specialization of the more general nonabelian tensor product introduced by Ronald Brown and Jean-Louis Loday (1987).

## Introduction

The investigation of this group construction from a group theoretic view started with a paper by Ronald Brown, David Johnson, and Edmund Robertson (1987).

The goals of their investigation include:

- Compute the nonabelian tensor square for a given $G$. That is, give a description of $G \otimes G$ that is simplified and easy to recognize.
- Determine the structure of the $G \otimes G$ from the structure of $G$.
- Compute homomorphisms of $G \otimes G$.

# Computing the tensor square

We have only the left conjugation action of $G$ to work with. This dictates what $G \otimes G$ is going to be.

For a finite group $G$, the definition gives us a finite presentation of $G \otimes G$. We can apply Tietze transformations to this presentation to obtain a simplified presentation of $G \otimes G$. We can then examine this simplified presentation to determine (in a more standard way) what the tensor square is.

Brown, Johnson, and Robertson (1987) compute the nonabelian tensor square of all nonabelian groups up to order 30 using Tietze transformations. This method does not scale well as we have $|G|^2$ generators and $2|G|^3$ relations.

## Rocco (1991)

Rocco (1991) considers the following group. Let $G$ and $G^\varphi$ be isomorphic groups via $\varphi : g \mapsto g^\varphi$, for all $g \in G$. Then

$$\nu(G) = \langle G, G^\varphi \mid {}^{k}[g, h^\varphi] = [{}^{k}g, ({}^{k}h)^\varphi] = {}^{k^\varphi}[g, h^\varphi], \forall g, h, k \in G \rangle,$$

Rocco investigates the structural aspects of $\nu(G)$ relative to $G$:

**Theorem 1.** *Let $G$ be a group.*

*(i) If $G$ is finite then $\nu(G)$ is finite.*

*(ii) If $G$ is a finite $p$-group then $\nu(G)$ is a finite $p$-group.*

*(iii) If $G$ is nilpotent of class $c$ then $\nu(G)$ is nilpotent of class at most $c + 1$.*

*(iv) If $G$ is solvable of derived length $d$ then $\nu(G)$ is solvable of class at most $d + 1$.*

# Ellis and Leonard (1995)

Independent of Rocco, Ellis defines the following group.

Let $G$ and $G^{\varphi}$ be isomorphic groups through $\varphi$, $\varphi : g \to g^{\varphi}$. Then consider the group

$$\chi(G) = (G * G^{\varphi})/\langle J \rangle$$

where $J = \{z[g, h^{\varphi}]z^{-1} \cdot [{}^{z}g, {}^{z}h^{\varphi}]^{-1} \mid z \in G * G^{\varphi}, g, h \in G\}$ is a normal generating set.

**Theorem 2.** *Let $G$ be a group. Then*

$$1 \to [G, G^{\varphi}] \xrightarrow{\iota} \chi(G) \xrightarrow{\xi} G \times G \to 1,$$

*is a short exact sequence where $\iota$ is the natural inclusion map and $\xi$ is the homomorphic extension of the map sending the generator $g \in G$ of $\chi(G)$ to $(g, 1)$ and the generator $g^{\varphi} \in G^{\varphi}$ of $\chi(G)$ to $(1, g)$.*

# Rocco, Ellis & Leonard

First $\chi(G)$ and $\nu(G)$ are the same groups.

The importance of this construction is pointed out in both papers:

**Theorem 3.** *Let $G$ be a group. The map $\sigma : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ is an isomorphism.*

Rocco was interested in the structure $\nu(G)$ and Ellis and Leonard in exploiting it to compute more effectively the nonabelian tensor products and squares.

If $G$ is a finite group with a presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ then it follows immediately using Rocco's presentation of $\nu(G)$ it has a finite presentation with $2|\mathcal{G}|$ generators and $2|\mathcal{R}| + 2|G|^3$ relations. Simpler than $G \otimes G$.

## Another Presentation for $\nu(G)$

For a finite group $G$, the presentation of $\nu(G)$ given in Ellis and Leonard reduces to $2|\mathcal{G}|$ generators and $2|\mathcal{R}| + |\mathcal{G}|^2 \cdot (c+t)$ relations where $c$ is the number of generators of $Z(G)$ the center of $G$ and $t = [G : Z(G)]$.

This a reasonably small presentation to compute and work with.

Hence to compute the nonabelian tensor square of $G$

1. Compute a presentation for $\nu(G)$.
2. Find a concrete representation for $\nu(G)$ ($p$-quotient algorithm, nilpotent quotient, coset enumeration).
3. Compute $[G, G^{\varphi}]$.

Ellis and Leonard used this method do compute the nonabelian tensor square for various groups such as $B(2,4)$ which has order $2^{12}$.

## Infinite Groups

In the Brown, Johnson, and Robertson paper, they define the concept of a cross pairing. Let $G$ and $L$ be groups. We call the mapping

$$\Phi : G \times G \to L$$

a crossed pairing if for all $g, g', g'' \in G$:

$$\begin{aligned} \Phi(gg', g'') &= \Phi({}^{g}g, {}^{g}g'')\Phi(g, g'') \\ \Phi(g, g'g'') &= \Phi(g, g')\Phi({}^{g'}g, {}^{g'}g''). \end{aligned}$$

This mapping lifts to a homomorphism $\Phi^* : G \otimes G \to L$ such that $\Phi^*(g \otimes g') = \Phi(g, g')$ for all $g, g' \in G$.

## Cross Pairings

Cross pairings give a method for computing the nonabelian tensor square of infinite groups.

1. Conjecture a group $L$.
2. Construct a cross pairing $\Phi : G \times G \to L$.
3. Show the $\Phi^*$ is an isomorphism.

This method has been used in computing the nonabelian tensor square of the

- free nilpotent groups of class 2, Bacon (1994);
- infinite metacyclic groups, Beuerle, Kappe (2000);
- free 2-Engel groups (groups satisfying the law $[x, y, y] = 1$) of finite rank, Bacon, Kappe, Morse (1997), Blyth, Morse, Redden (2004).

## Free 2-Engel Groups

The cross pairing pairing approach requires considerable insight into the groups $L$ and $G$ in order to construct $\Phi$ and check that it is a cross pairing.

When $G \otimes G$ is abelian, checking whether a conjectured map $\Phi$ is a cross pairing is relatively straightforward as with the nilpotent of class 2 groups and the infinite metacyclic groups.

When $G \otimes G$ is not abelian the computations involved in computing the tensor square become overwhelmingly complex and the appropriate checks that $\Phi$ is a cross pairing are possible only with computer assistance.

The following slide is from our paper on 2-Engel groups in the Proceeding of the Edinburgh Mathematical Society.

## Free nilpotent groups of class 3

Computing the nonabelian tensor square of the free 2-Engel groups was supposed to be the precursor to computing the tensor square of the free nilpotent groups of class 3.

The complexity involved with the cross pairing was overwhelming. It was obvious a different approach was needed.

We revisited the work of Rocco and Ellis to see if any of it could be of any practical use for infinite groups.

## $G \otimes G$, $\nu(G)$ and polycyclic groups

The following exact sequences are found in Brown, Johnson, and Robertson (1987):

$$0 \longrightarrow J_2(G) \longrightarrow G \otimes G \xrightarrow{\kappa} G' \longrightarrow 1$$

where $\kappa(g \otimes h) = [g, h]$,

$$H_3(G) \longrightarrow \Gamma(G_{ab}) \xrightarrow{\psi} J_2(G) \xrightarrow{\phi} H_2(G) \longrightarrow 0$$

where $G_{ab} = G/G'$.

The following short exact sequence is due to Ellis:

$$1 \longrightarrow [G, G^{\varphi}] \xrightarrow{\iota} \nu(G) \xrightarrow{\xi} G \times G \longrightarrow 1.$$

**Theorem 4.** *Suppose that $G$ is a polycyclic group. Then both $\nu(G)$ and $G \otimes G$ are also polycyclic.*

# Observations

All “tensor” computations can be done as commutator calculations. We truly have a commutator connection. The following lemmas are due to Rocco (1991).

**Lemma 5.** *The following relations hold in* $\nu(G)$*:*

*(i)* ${}^{[g_3, g_4^{\varphi}]}[g_1, g_2^{\varphi}] = {}^{[g_3, g_4]}[g_1, g_2^{\varphi}]$ *and*
${}^{[g_3^{\varphi}, g_4]}[g_1, g_2^{\varphi}] = {}^{[g_3, g_4]}[g_1, g_2^{\varphi}]$ *for all* $g_1, g_{2,}, g_3, g_4 \in G$*;*

*(ii)* $[g_1^{\varphi}, g_2, g_3] = [g_1, g_2, g_3^{\varphi}] = [g_1^{\varphi}, g_2, g_3^{\varphi}]$ *and*
$[g_1, g_2^{\varphi}, g_3] = [g_1^{\varphi}, g_2^{\varphi}, g_3] = [g_1, g_2^{\varphi}, g_3^{\varphi}]$ *for all* $g_1, g_{2,}, g_3 \in G$*;*

*(iii)* $[g, g^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g \in G$*;*

*(iv)* $[g_1, g_2^{\varphi}][g_2, g_1^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g_1, g_2 \in G$*;*

*(v)* $[g, g^{\varphi}] = 1$ *for all* $g \in G'$*.*

# Observations (cont)

**Lemma 6.** *Let* $a, b$ *and* $x$ *be elements of* $G$ *such that* $[x, a] = 1 = [x, b]$*. Then in* $\nu(G)$*,*

$$[a, b, x^{\varphi}] = 1 = [[a, b]^{\varphi}, x].$$

**Lemma 7.** *Let* $x$ *and* $y$ *be elements of* $G$ *such that* $[x, y] = 1$*. Then in* $\nu(G)$*,*

*(i)* $[x^n, y^{\varphi}] = [x, y^{\varphi}]^n = [x, (y^{\varphi})^n]$ *for all integers* $n$*;*

*(ii) If* $x$ *and* $y$ *are torsion elements of orders* $o(x)$ *and* $o(y)$ *in* $G$*, then the order of* $[x, y^{\varphi}]$ *in* $\nu(G)$ *divides the greatest common divisor of* $o(x)$ *and* $o(y)$*.*

We have nilpotency and solvability bounds on $\nu(G)$ which make these computations easier.

We can write down a generating set for $G \otimes G$ relative to the polycyclic generating sequence of $G$.

## Observations (cont)

**Corollary 8.** *Let $G$ be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then the subgroup $[G, G^{\varphi}]$ of $\nu(G)$ is generated by $\{[\mathfrak{g}_i, \mathfrak{g}_i^{\varphi}], [\mathfrak{g}_i^{\epsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}]\}$ for $1 \leq i, j, \leq k$, $i \neq j$ where*

$$\epsilon = \begin{cases} 1, & \text{if } |\mathfrak{g}_i| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_i| = \infty \end{cases} \qquad \delta = \begin{cases} 1, & \text{if } |\mathfrak{g}_j^{\varphi}| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_j^{\varphi}| = \infty. \end{cases}$$

As an immediate application we are able to obtain a rough statement about a class of groups which include the infinite metacyclic groups.

**Corollary 9.** *Let $G$ be a polycyclic group with a polycyclic generating sequence of length 2. Then $G \otimes G$ is abelian and is the direct product of at most 4 cyclic groups.*

# A theorem of A. McDermott

Recall we have a subgroup $\langle J\rangle$ which was finitely generated when $G$ was finite. Ellis had a good bound on the number of elements of $J$. McDermott in computing the tensor product for finite groups wanted to improve this bound to improve the efficiency of his code.

If one looks closely at the proof and the statement of his theorem we get the following special case for polycyclic groups.

**Theorem 10.** *Let $G$ be a polycyclic group. Then the subgroup $\langle J\rangle$ is normally generated by the words*

$$\mathfrak{g}[g_i, g_j^{\varphi}]\mathfrak{g}^{-1}[{}^{\mathfrak{g}}g_i, ({}^{\mathfrak{g}}g_j)^{\varphi}]^{-1} \quad \textit{and} \quad \mathfrak{g}^{\varphi}[g_i, g_j^{\varphi}](\mathfrak{g}^{\varphi})^{-1}[{}^{\mathfrak{g}}g_i, ({}^{\mathfrak{g}}g_j)^{\varphi}]^{-1}$$

*where $g_i, g_j$ are elements of $\mathcal{G}$ any generating set for $G$ and $\mathfrak{g}$ is an element of $\mathfrak{G}$ a polycyclic generating sequence for $G$.*

The consequence of this is we have a finite presentation for $\nu(G)$.

## Computer implementations

Given a polycyclic group $G = \langle \mathcal{G} \mid \mathcal{R} \rangle$ and a polycyclic generating sequence $\mathfrak{G}$. We can write down a finite presentation for $\nu(G)$

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^{\varphi} \mid \mathcal{R}, \mathcal{R}^{\varphi}, J \rangle$$

where $J$ is determined by McDermott's result.

We know $\nu(G)$ is polycyclic. So we can apply a polycyclic quotient algorithm to find a polycyclic presentation for $\nu(G)$.

Current implementations of the polycyclic quotient algorithms fail to terminate even for the simplest case – the infinite dihedral group $D_0$.

The group $\nu(D_0)$ has 4 generators and 18 relations.

However, if $G$ is nilpotent we can use the nilpotent quotient algorithm. Moreover we have a class bound on $\nu(G)$ and can compute this quotient even faster.

## Algorithm for Nilpotent Groups

Let $G = \langle \mathcal{G} \mid \mathcal{R} \rangle$ be a nilpotent group of class $c$ with a polycyclic generating sequence $\mathfrak{G}$. Then we have the following algorithm.

1. Write out a finite presentation of $\nu(G)$ using McDermott's theorem.
2. Compute a polycyclic presentation for $\nu(G)$ using a nilpotent quotient algorithm.
3. Compute $[G, G^{\varphi}]$ (which can be effectively computed for any polycyclic group (infinite or finite)).

The subgroup $[G, G^{\varphi}]$ gives a polycyclic presentation for $G \otimes G$.

Hence we can use the computer to compute examples of the free 2-Engel groups with small rank, and free nilpotent groups of small rank and class. This was implemented in GAP using the "nq" and "Polycyclic" packages.

## More Application

As an application of the earlier theoretical results using guidance from computational results for small rank we compute the nonabelian tensor square of the free nilpotent groups of class 3 and finite rank.

All calculations involve working with the commutators in $[G, G^{\varphi}]$.

The problem involves finding exact structure of $[G, G^{\varphi}]$.

Using cross pairings, the “simple” free 2-Engel case of rank $n$ took 2 published papers totaling 33 pages plus a 117 page dissertation + 42 pages of supporting materials.

The free nilpotent class 3 rank $n$ will be written up in about 7 published pages which includes all details.

## Free nilpotent groups of class 3

**Theorem 11.** *Let $G$ be a free nilpotent group of class 3 and rank $n$. Then $G \otimes G \cong N \times A$ where $N$ is nilpotent of class 2 with rank $n(n-1)$ and $A$ is free abelian of rank $f(n)$ where*

$$\begin{aligned} f(n) &= n + 2\binom{n}{3} + 2\binom{n}{2} + 6\binom{n}{3} + 3\binom{n}{4} + 3\binom{n}{2} \\ &= n + 5\binom{n}{2} + 8\binom{n}{3} + 3\binom{n}{4} \\ &= \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}. \end{aligned}$$

## Capable Groups

Another application of computing the nonabelian tensor square of a group is determining if a group is capable.

A group $G$ is **capable** if there exists a group $H$ such that $G \cong H/Z(H)$.

The epicenter $Z^*(G)$ of a group $G$ is the intersection of all the central extensions of $G$.

**Theorem 12 (Beyl, Felgner, Schmid).** *A group $G$ is capable if and only if $Z^*(G) = 1$.*

## Exterior Square and Center of a group

Let $\nabla(G)$ denote the central subgroup of $G \otimes G$ generated by the elements $x \otimes x$ for $x \in G$.

Then $G \wedge G = (G \otimes G)/\nabla(G)$.

We define the exterior center of a group $G$ as

$$Z^{\wedge}(G) = \{a \in G \mid a \wedge g = 1_{\wedge}, \forall g \in G\}.$$

$Z^{\wedge}(G)$ is subgroup of the center of $G$.

**Theorem 13 (Ellis).** *For any group $G$, the epicenter coincides with the exterior center, i.e. $Z^*(G) = Z^{\wedge}(G)$. Hence a group $G$ is capable if and only if $Z^{\wedge}(G) = 1_G$.*

## Determining Capability

Let $T$ be the subgroup of $[G, G^{\varphi}]$ generated by $[g, g^{\varphi}]$. Then $T$ is isomorphic to $\nabla(G)$. Hence $[G, G^{\varphi}]/T$ is isomorphic to $G \wedge G$.

We note that $T$ is central in $\nu(G)$.

Let $\tau(G) = \nu(G)/T$. Then $[G, G^{\varphi}]$ in $\tau(G)$ is isomorphic to $G \wedge G$.

We have the $[G, \langle a \rangle^{\varphi}] = 1$ if and only if $a$ is an element of $Z^{\wedge}(G)$.

Because of the lack of torsion in the free nilpotent groups, we can't see how $[G, \langle a \rangle^{\varphi}] = 1$ for any nontrivial $a$. Hence we conjecture that such groups are all capable.