Computing the nonabelian tensor square of polycyclic groups

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Abstract

We provide a method for computing the nonabelian tensor square for any polycyclic group. We provide an implementation of this method for finitely generated nilpotent groups and use it to compute the nonabelian tensor square of the free nilpotent of class 3 groups of rank n.

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1 Introduction

The nonabelian tensor square $G \otimes G$ of the group G is the group generated by the symbols $g \otimes h$, where $g, h \in G$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all $g, g, h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$ is conjugation on the left.

By computing the nonabelian tensor square we mean finding a standard or simplified presentation for the nonabelian tensor square. One approach to computing the nonabelian tensor square for a finite group, used by Brown, Johnson and Robertson [5], is to start with the finite presentation given by the definition above and simplify the presentation using Tietze transformations. The simplified presentation is then examined to determine the isomorphism type of the nonabelian tensor square. This method was used in [5] to compute the nonabelian tensor square for each group of order at most 30. Since the presentation given by the definition of $G \otimes G$ has $|G|^2$ generators and $2 \cdot |G|^3$ relations, this method is limited to groups of relatively small order.

A second approach to computing the nonabelian tensor square of a finite group involves the group $\nu(G)$ whose presentation given below is due to Rocco [15]. Let G and G^φ be isomorphic groups via $\varphi : g \mapsto g^\varphi$ for all $g \in G$. Define the group $\nu(G)$ to be

$$\nu(G) = \langle G, G^\varphi \mid {}^k[g, h^\varphi] = [{}^k g, \left({}^k h\right)^\varphi] = {}^{k^\varphi}[g, h^\varphi] \text{ for all } g, h, k \in G \rangle,$$

where ${}^x y = xyx^{-1}$ is conjugation on the left and the commutator $[x, y]$ is defined to be $xyx^{-1}y^{-1}$. The group $\nu(G)$ was also introduced independently of Rocco by Ellis and Leonard in [10] using a different presentation. The motivation for studying this group is the following result found in both papers [15] and [10].

Theorem 1 *Let G be a group. The map $\tau : G \otimes G \to [G, G^\varphi] \lhd \nu(G)$ defined by $\tau(g \otimes h) = [g, h^\varphi]$ is an isomorphism.*

Rocco's focus in [15] was on obtaining structural results for $\nu(G)$ relative to those of G. He only hints that one may be able to compute the nonabelian tensor square effectively within this larger group. The following theorem summarizes several of Rocco's results.

Theorem 2 *Let G be a group.*

(i) If G is finite then $\nu(G)$ is finite.
(ii) If G is a finite p-group then $\nu(G)$ is a finite p-group.
(iii) If G is nilpotent of class c then $\nu(G)$ is nilpotent of class at most $c + 1$.
(iv) If G is solvable of derived length d then $\nu(G)$ is solvable of class at most $d + 1$.

Ellis and Leonard [10] start with an infinite presentation of $\nu(G)$ and show for a finite group that a relatively small presentation can be given for $\nu(G)$. The primary purpose of their paper was to compute the nonabelian tensor square and the more general nonabelian tensor product. They provide several general structural results not found in Rocco's paper [15].

Theorem 3 ([10]) *Let G be a group. Let $\iota : [G, G^\varphi] \to \nu(G)$ be the natural inclusion map and let $\xi : \nu(G) \to G \times G$ be the homomorphic extension of the map sending the generator $g \in G$ of $\nu(G)$ to $(g, 1)$ and the generator $g^\varphi \in G^\varphi$ of $\nu(G)$ to $(1, g)$. Then*

$$1 \to [G, G^\varphi] \xrightarrow{\iota} \nu(G) \xrightarrow{\xi} G \times G \to 1, \qquad (1)$$

is a short exact sequence.

One could use Theorem 3 to compute $[G, G^{\varphi}]$. This requires finding, for example by coset enumeration, a concrete representation of $\nu(G)$, setting up the homomorphism ξ, and computing the kernel of ξ. However, Ellis and Leonard did not have available the structural results of Rocco that guarantee that $\nu(G)$ is finite whenever G is. By a result of [5], Ellis and Leonard knew that if G is a finite p-groups then $G \otimes G$ is a finite p-group. It follows in this case that the image of $[G, G^{\varphi}]$ within the largest finite p-quotient of $\nu(G)$ is isomorphic to $G \otimes G$. (Of course, by Theorem 2 the largest finite p-quotient of $\nu(G)$ is just $\nu(G)$ itself.)

I wanted to tighten the reference here - on page 139 E and L say that this follows by their Theorem 1 and Ellis's JA paper. What reference is the right one here?

To facilitate computing $[G, G^{\varphi}]$, Ellis and Leornard prove the following theorem.

Theorem 4 ([10]) *Let G be a group. Then*

$$[G, G^{\varphi}] = G^{\nu(G)} \cap (G^{\varphi})^{\nu(G)}.$$

The algorithm used in [10] to compute the nonabelian tensor square of a finite p-group involves finding a finite presentation of $\nu(G)$, computing $\nu_p(G)$ (the largest p-quotient of $\nu(G)$), computing the normal closures of G_p and G_p^{φ} in $\nu_p(G)$, and, finally, computing the intersection of these two normal closures. Using this algorithm Ellis and Leonard computed the nonabelian tensor squares of selected 2- and 3-groups, for example, the Burnside group $B(2, 4)$ of exponent 4 and order 4096.

In general, if G is a finite group with a presentation $\langle \mathcal{G} \mid R \rangle$, then Rocco's presentation of $\nu(G)$ has $2|\mathcal{G}|$ generators and $2|R| + 2|G|^3$ relations. The presentation given in [10] reduces the number of relations to $|\mathcal{G}|^2 \cdot (c + t)$, where c is the number of generators of the center $Z(G)$ of G and t is the index of $Z(G)$ in G. Further reductions in the number of relations in a presentation for $\nu(G)$ for finite groups were made by McDermott [13]. Computer implementations for computing the nonabelian tensor square (and other homological functors) using the results mentioned here were given in [10] using CAYLEY, in Ellis [9] using Magma, and by McDermott [13] using GAP [11].

Should we give a reference for Magma?

Until now, the only tool for computing the nonabelian tensor square for infinite groups was the cross pairing. Let G and L be groups. A function $\Phi : G \times G \to L$ is called a *crossed pairing* if

$$\Phi(gh, k) = \Phi({}^{g}h, {}^{g}k)\Phi(g, k) \text{ and } \Phi(g, hk) = \Phi(g, h)\Phi({}^{h}g, {}^{h}k)$$

for all $g, h, k \in G$.

The following proposition allows us to determine homomorphic images of

$G \otimes G$.

Proposition 5 ([6]) *A crossed pairing $\Phi : G \times G \to L$ determines a unique homomorphism of groups $\Phi^* : G \otimes G \to L$ such that $\Phi^*(g \otimes g') = \Phi(g, g')$ for all g, g' in G.*

To compute the tensor square of a group G using this method, one conjectures a group L that is believed to be isomorphic to $G \otimes G$. A crossed pairing Φ is constructed and then it is shown that the associated map Φ^* is an isomorphism. This approach requires considerable insight into the groups L and G in order to construct Φ and check that it is a cross pairing. When $G \otimes G$ is abelian, checking whether a conjectured map Φ is a cross pairing is relatively straightforward. For example, Bacon [1] computed the nonabelian tensor square for the free nilpotent of class 2 groups of rank n using this approach. When $G \otimes G$ is not abelian the computations involved in computing the tensor square become overwhelmingly complex and the appropriate checks that Φ is a crossed pairing are possible only with computer assistance. This complexity can be seen in Bacon, Kappe and Morse [2] and Blyth, Morse and Redden [4], in which the nonabelian tensor squares of the free 2-Engel groups of rank n are computed. Insight into conjecturing L for these groups was aided by computing for small rank n the nonabelian tensor square of the Burnside group of exponent 3 and rank n, which is a finite homomorphic image of the free 2-Engel group of rank n. These computations were performed using the methods of Ellis and Leonard [10]. We arrived at the following result:

Theorem 6 ([4]) *The nonabelian tensor square of the free 2-Engel group of rank $n > 2$ is a direct product of a free abelian group of rank $\frac{n(n^2+2)}{3}$ and an $n(n-1)$-generated nilpotent group of class 2 whose derived subgroup has exponent 3.*

The original goal of this paper was to compute the nonabelian tensor squares of the free nilpotent groups of class 3 and rank n. Based on the complexity involved in computing the nonabelian tensor squares of the quotient class of free 2-Engel groups, it became clear that we needed a different approach to the problem.

In this paper we use and extend the work of Ellis and Leonard [10] and of Rocco [15] to compute the nonabelian tensor squares of groups in a significantly larger class, namely the class of polycyclic groups. In Section 2 we record and reconcile the independent work of Rocco [15] and Ellis and Leonard [10] and fix the notation used throughout the paper. In Section 3 we prove our first theorem, the proof of which uses the structural results from both [10] and [15].

Theorem A *Suppose that G is a polycyclic group. Then both $\nu(G)$ and $G \otimes G$ are also polycyclic.*

For a polycyclic group G we are then able to determine a finite generating set for $[G, G^{\varphi}]$ in terms of a polycyclic generating sequence for G (Corollary 21). Moreover, if we know the derived length or the nilpotency class of G then by Theorem 2 we know the derived length or nilpotency class of $\nu(G)$. Hence we can determine a significant amount about $G \otimes G$ by using commutator calculus within $[G, G^{\varphi}]$ and the identities of Rocco found in Section 2. As a simple application of Theorem A and Corollary 21, we completely compute the nonabelian tensor square of the infinite dihedral group toward the end of Section 3.

Since $\nu(G)$ is polycyclic it has a finite presentation. In Section 4, by specializing Theorem 2.2.8 of McDermott [13], we construct a finite presentation of $\nu(G)$ using a polycyclic generating sequence for G and a generating set $\mathcal{G}$ of G. Hence we can use a polycyclic quotient algorithm to find a polycyclic presentation for $\nu(G)$. Once we have a polycyclic presentation for $\nu(G)$, computing $[G, G^{\varphi}]$ within a polycyclic group becomes computationally feasible (see, for example, [8]). Polycyclic quotient algorithms exist and are implemented — see, for example, [12] and [7]. However, even for the simplest infinite non nilpotent group, namely the infinite dihedral group, these polycyclic quotient algorithms are unable to find a polycyclic presentation for $\nu(G)$ and hence $[G, G^{\varphi}]$ cannot be computed at present using this method.

Fortunately, when G is a finitely generated nilpotent group, both $G \otimes G$ and $\nu(G)$ are not only polycyclic but, by Theorem A and Theorem 2, are also finitely generated nilpotent groups. Efficient implementations of the nilpotent quotient algorithm exist that make it possible to compute the nonabelian tensor squares of the free nilpotent groups of class c and rank n. In particular, we use the nq [14] package in GAP [11] and its supporting package Polycyclic [8] to determine $G \otimes G$ when G is a finitely generated nilpotent group. In practice, using a fast workstation we are able to compute the nonabelian tensor squares of the free nilpotent groups of class $c < 10$ and rank $n < 7$. For other classes of finitely generated nilpotent groups (for example, the free 2-Engel groups of finite rank), we are able to compute nonabelian tensor squares for higher nilpotence class and rank.

We conclude the paper by applying our new results to our original goal of computing the nonabelian tensor square of the free nilpotent groups of class 3 and finite rank. Our analysis was aided by direct computations made for small rank using the algorithm in Section 4.

Theorem B *Let G be a free nilpotent group of class 3 and rank n. Then $G \otimes G \cong N \times A$, where N is nilpotent of class 2 with rank $n(n-1)$ and A is free abelian of rank $f(n)$.*

2 Technical Preliminaries

In this section we formalize and standardize our notation particularly the use of left conjugation and its consequences in the commutator calculus. We then integrate the independent work of Ellis and Leonard [], McDermott [] and Rocco [].

The following four lemmas are due to Rocco [15].

Lemma 7 *The following relations hold in $\nu(G)$:*

(i) ${}^{[g_3,g_4^{\varphi}]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *and*
${}^{[g_3^{\varphi},g_4]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *for all* $g_1, g_{2,}, g_3, g_4 \in G$*;*
(ii) $[g_1^{\varphi}, g_2, g_3] = [g_1, g_2, g_3^{\varphi}] = [g_1^{\varphi}, g_2, g_3^{\varphi}]$ *and*
$[g_1, g_2^{\varphi}, g_3] = [g_1^{\varphi}, g_2^{\varphi}, g_3] = [g_1, g_2^{\varphi}, g_3^{\varphi}]$ *for all* $g_1, g_{2,}, g_3 \in G$*;*
(iii) $[g, g^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g \in G$*;*
(iv) $[g_1, g_2^{\varphi}][g_2, g_1^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g_1, g_2 \in G$*;*
(v) $[g, g^{\varphi}] = 1$ *for all* $g \in G'$*.*

Lemma 8 *Let x_i, y_i be elements of G. For $z = \prod_{i=1}^{s}[x_i, y_i]$, where $x_i, y_i \in G$ for $i = 1, \dots s$, we define $\tilde{z}$ to be $\prod_{i=1}^{s}[x_i^{\varphi}, y_i]$ the following identities hold in $\nu(G)$:*

(i) ${}^{[g_1^{\varphi},g_2]}\tilde{z} = {}^{[g_1,g_2]}\tilde{z}$ *for all* $g_1, g_2 \in G$*;*
(ii) ${}^{\tilde{z}}[g_1, g_2^{\varphi}] = {}^{z}[g_1, g_2^{\varphi}]$ *for all* $g_1, g_2 \in G$*;*
(iii) $[z, g^{\varphi}] = [\tilde{z}, g]$ *for all* $g \in G$*.*

Lemma 9 *Let a, b and x be elements of G such that $[x, a] = 1 = [x, b]$. Then in $\nu(G)$,*

$$[a, b, x^{\varphi}] = 1 = [[a, b]^{\varphi}, x].$$

Lemma 10 *Let x and y be elements of G such that $[x, y] = 1$. Then in $\nu(G)$,*

(i) $[x^n, y^{\varphi}] = [x, y^{\varphi}]^n = [x, (y^{\varphi})^n]$ *for all integers n;*
(ii) If x and y are torsion elements of orders $o(x)$ and $o(y)$ in G, then the order of $[x, y^{\varphi}]$ in $\nu(G)$ divides the greatest common divisor of $o(x)$ and $o(y)$.

Our first concern showing that the group

From Rocco's *Communications in Algebra* paper which uses conjugation on the right for the commutators we have the following group construction.

Given groups G and G^{φ}, isomorphic through an isomorphism $\varphi : G \to G^{\varphi}$,

$g \to g^\varphi$ for all $g \in G$, then we define the group

$$\nu(G) = \langle G, G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^{k^\varphi}, \forall g, h, k \in G\rangle,$$

that is, $\nu(G)$ is the quotient of the free product $G * G^\varphi$ by its normal subgroup generated by all the words $[g, h^\varphi]^k \cdot [g^k, (h^k)^\varphi]^{-1}$ and $[g, h^\varphi]^{k^\varphi} \cdot [g^k, (h^k)^\varphi]^{-1}$ for all $g, h, k \in G$.

Substituting g^{-1}, h^{-1} and k^{-1} for g, h, k above we get an equivalent version with conjugation on the left i.e. ${}^k h = khk^{-1}$ and $[g, h] = ghg^{-1}h^{-1}$. Hence we can state

The group $\nu(G)$ is the quotient of the free product $G * G^\varphi$ by its normal subgroup generated by all the words ${}^k[g, h^\varphi] \cdot [{}^k g, ({}^k h)^\varphi]^{-1}$ and ${}^{k^\varphi}[g, h^\varphi] \cdot [{}^k g, ({}^k h)^\varphi]^{-1}$ for all $g, h, k \in G$. We denote this set of words by I. That is,

$$I = \{{}^k[g, h^\varphi] \cdot [{}^k g, ({}^k h)^\varphi]^{-1}, {}^{k^\varphi}[g, h^\varphi] \cdot [{}^k g, ({}^k h)^\varphi]^{-1} \mid g, h, k \in G\}.$$

From now on all commutators will be defined with left conjugation. Ellis and Leonard's group can be defined as the following:

Let G and G^φ be isomorphic groups through φ, $\varphi : g \to g^\varphi$. Then consider the group

$$(G * G^\varphi)/\langle J\rangle$$

where

$$J = \{z[g, h^\varphi]z^{-1} \cdot [{}^z g, {}^z h^\varphi]^{-1} \mid z \in G * G^\varphi, g, h \in G\}$$

is a normal generating set.

Our goal is to show that

$$\nu(G) \cong (G * G^\varphi)/\langle J\rangle.$$

We will do so by a series of claims. The first it to show explicitly what it means for J to be a normal generating set.

Claim 11 *The set J is a normal generating set. That is for any element u of J and any element w of $G * G^\varphi$ we have ${}^w u$ a product of elements of J.*

Proof. Let $u = z[g, h^\varphi]z^{-1} \cdot [{}^z g, {}^z h^\varphi]^{-1}$ be an arbitrary element of J and w an arbitrary element of $G * G^\varphi$. Then

$$\begin{aligned}
{}^w(z[g, h^\varphi]z^{-1} \cdot [{}^z g, {}^z h^\varphi]^{-1}) &= wz[g, h^\varphi]z^{-1}w^{-1} \cdot {}^w[{}^z g, {}^z h^\varphi]^{-1} \\
&= wz[g, h^\varphi]z^{-1}w^{-1} \cdot [{}^{wz} g, {}^{wz} h^\varphi]^{-1} \cdot [{}^{wz} g, {}^{wz} h^\varphi] \cdot {}^w[{}^z g, {}^z h^\varphi]^{-1} \\
&= wz[g, h^\varphi]z^{-1}w^{-1} \cdot [{}^{wz} g, {}^{wz} h^\varphi]^{-1} \cdot [{}^w({}^z g), {}^w({}^z h)^\varphi] \cdot {}^w[{}^z g, {}^z h^\varphi]^{-1} \\
&= wz[g, h^\varphi]z^{-1}w^{-1} \cdot [{}^{wz} g, {}^{wz} h^\varphi]^{-1} \cdot \left(w[{}^z g, {}^z h^\varphi]w^{-1} \cdot [{}^w({}^z g), {}^w({}^z h)^\varphi]^{-1}\right)^{-1} \\
&= UV^{-1}
\end{aligned}$$

It is clear that U and V are elements of J as needed. $\square$

Lemma 12 *The subgroup $\langle J\rangle = J^{G*G^\varphi}$, the normal closure of J in $G*G^\varphi$.*

Proof. An arbitrary element of J^{G*G^φ} can be expressed as

$$^{w_1}u_1^{\pm1}\cdots{}^{w_n}u_n^{\pm1} = (^{w_1}u_1)^{\pm1}\cdots(^{w_n}u_n)^{\pm1}$$

where the u_i are elements of J and the w_i are elements of $G*G^\varphi$. But by Claim 11 each $^{w_i}u_i$ is a product of elements in $J\cup J^{-1}$. $\square$

Lemma 13 *The set I is a subset of J.*

Proof. Let k be an arbitrary element of G. Since k and k^φ are elements of $G*G^\varphi$, both words $^k[g,h^\varphi]\cdot[^kg,(^kh)^\varphi]^{-1}$ and $^{k^\varphi}[g,h^\varphi]\cdot[^kg,(^kh)^\varphi]^{-1}$ are in J. $\square$

Lemma 14 *The subgroup $I^{G*G^\varphi} = \langle J\rangle$.*

Proof. It follows from Claim 13 that I^{G*G^φ} is contained in $\langle J\rangle$. To show double containment we will show that every element in J can we written as a product of conjugates of elements in I.

Let $u = z[g,h^\varphi]z^{-1}\cdot[^zg,{}^zh^\varphi]^{-1}$ be an arbitrary element of J for some $z\in G*G^\varphi$ and $g,h\in G$. If $z\in G$ or $z\in G^\varphi$ then u is an element of I and we are done. Let z an arbitrary element of $G*G^\varphi$. Then z is uniquely expressed as a reduced sequence $x_1x_2\cdots x_r$ where the x_i are elements of the base groups G or G^φ and x_i and x_{i+1} are not both in the same base group. Without loss of generality suppose that $x_r = y^\varphi$ is in G^φ and abbreviate $z = xy^\varphi$. Then using the identity from Lemma 10 in the paper we have

$$x(y^\varphi[g,h^\varphi]y^{-\varphi}[^yg,{}^yh^\varphi]^{-1})x^{-1}x[g,h^\varphi]x^{-1}[^xg,{}^xh^\varphi]^{-1} = xUx^{-1}V$$

where U is an element of I. Now $x = x_1\cdots x_{r-1}$ where x_{r-1} is in the base group G and we can repeat the process on V. Continuing in this fashion to x_1 we see that u, our arbitary element of J, is a product of conjugates of I as needed. $\square$

3 Computing $G\otimes G$ for G polycyclic

In this section we show that if G is polycyclic then $G\otimes G$ is polycyclic. It follows that $G\otimes G$ has a finite presentation. The question then becomes finding this presentation. To this end we will show that $\nu(G)$ is polycyclic whenever G is. Since $\nu(G) = \langle G, G^\varphi\rangle$ where G and G^φ are subgroups of $\nu(G)$, we can then explicitly write down the finite generators of $[G,G^\varphi]\le\nu(G)$ which is isomorphic to $G \otimes G$. We conclude this section with an application of these results to groups with a polycyclic generating sequence of length 2 which includes computing the nonabelian tensor square of the infinite dihedral group.

The following exact sequence is found in [5]:

$$0 \longrightarrow J_2(G) \longrightarrow G \otimes G \xrightarrow{\kappa} G' \longrightarrow 1 \tag{2}$$

where $\kappa(g \otimes h) = [g, h]$.

The kernel of κ, $J_2(G)$, is an central subgroup of $G \otimes G$ by [5]. Suppose that G is a polycylic group. Then G' is polycyclic. If we can show that $J_2(G)$ is finitely generated then it is polycylic since a finitely generated abelian group is polycyclic. It follows that $G \otimes G$ is an extension of polycyclic groups and hence polycyclic. The following exposition is needed to prove show that $J_2(G)$ is finitely generated.

Given an abelian group A, the Whitehead universal quadratic functor ΓA is the abelian group with generators γa, $a \in A$, and defining relations

$$\gamma(a^{-1}) = \gamma a \quad \text{and} \quad \gamma(abc)\gamma a\gamma b\gamma c = \gamma(ab)\gamma(bc)\gamma(ca)$$

for all $a, b, c \in A$. From the definition the following properties hold (see [5]):

Lemma 15 *Let A and B be abelian groups. Then*

$$\Gamma(A \times B) \cong \Gamma A \times \Gamma B \times (A \otimes B)$$

$$\Gamma \mathbb{Z}_n \cong \begin{cases} \mathbb{Z}_n, & n \text{ odd} \\ \mathbb{Z}_{2n}, & n \text{ even} \end{cases}$$

where $\mathbb{Z}_n = \langle x \mid x^n = 1 \rangle$ for $n \geq 0$ taking $\mathbb{Z}_0 = \mathbb{Z}$, the infinite cyclic group. Here $A \otimes B$ is the ordinary tensor product.

The following corollary is a direct consequence of Lemma 15.

Corollary 16 *Let A be a finitely generated abelian group. Then ΓA is finitely generated.*

PROOF. The A be a finitely generated abelian group. Hence $A \cong A_1 \times \cdots \times A_n$ where each A_i is cyclic. We induct on n. For $n = 1$ we have ΓA_1 is cyclic by Lemma 15. Therefore it is finitely generated by needed. Suppose the result is true for $n - 1 \geq 1$. Then by Lemma 15

$$\Gamma A = \Gamma((A_1 \times \cdots \times A_{n-1}) \times A_n) = \Gamma(A_1 \times \cdots \times A_{n-1}) \times \Gamma A_n \times ((A_1 \times \cdots \times A_{n-1}) \otimes A_n).$$

By the induction hypothesis $\Gamma(A_1 \times \cdots \times A_{n-1})$ is finitely generated as is the other terms of the product. Hence ΓA is finitely generated as needed.

We denote the nth dimensional integral homology group of a group G by $H_n(G)$. In [5] we have the following exact sequence:

$$H_3(G) \longrightarrow \Gamma(G_{ab}) \xrightarrow{\psi} J_2(G) \xrightarrow{\phi} H_2(G) \longrightarrow 0 \qquad (3)$$

where $G_{ab} = G/G'$.

Lemma 17 *Let G be a finitely presented group such that G_{ab} is finitely generated. Then $J_2(G)$ is finitely generated.*

PROOF. Since G is finitely presented then $H_2(G)$ is a finitely generated abelian group [16] and hence polycyclic. By hypothesis, G_{ab} is finitely generated so it follows from Lemma 15 ΓG_{ab} is finitely generated and it too is a polycyclic group. Since the sequence (3) is exact, the image of ψ is equal to the kernel of ϕ and hence kernel of ϕ is polycyclic. Therefore $J_2(G)$ is an extension of two polycyclic groups. It follows $J(G)$ is polycyclic. So it is finitely generated as needed.

Corollary 18 *Let G be a polycyclic. Then $J_2(G)$ is finitely generated.*

PROOF. Every polycyclic group is finitely presented. Moreover G_{ab} is polycyclic and hence finitely generated. The result now follows from the Lemma 17.

Proposition 19 *Let G be a polycyclic group. Then $G \otimes G$ is polycyclic.*

PROOF. By Corollary 18 the group $J_2(G)$ is polycyclic. By the exact sequence (2) we see that $G \otimes G$ is an extension of two polycyclic groups. Hence $G \otimes G$ is polycyclic as needed.

Corollary 20 *If G is polycyclic then $\nu(G)$ is polycyclic.*

PROOF. Let G be a polycyclic group. Then both $G \times G$ and $[G, G^{\varphi}] \cong G \otimes G$ are polycyclic. The latter by Proposition 19. Hence $\nu(G)$ is an extension of two polycyclic groups by Theorem 3 and therefore is polycyclic.

This result about $\nu(G)$ allows us to explicitly write down a finite generating set for $[G.G^{\varphi}]$ in terms of a polcyclic generating set of G.

Corollary 21 *Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then the subgroup $[G, G^\varphi]$ of $\nu(G)$ is generated by $\{[\mathfrak{g}_i, \mathfrak{g}_i^\varphi], [\mathfrak{g}_i^\epsilon, (\mathfrak{g}_j^\varphi)^\delta]\}$ for $1 \leq i, j, \leq k$, $i \neq j$ where*

$$\epsilon = \begin{cases} 1, & \text{if } |\mathfrak{g}_i| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_i| = \infty \end{cases} \qquad \delta = \begin{cases} 1, & \text{if } |\mathfrak{g}_j^\varphi| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_j^\varphi| = \infty. \end{cases}$$

The following known result about polycyclic groups almost completes the proof of Corollary 21.

Lemma 22 *Let H be a polycyclic group with subgroups A and B with polycyclic generating sets $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_m$ respectively. If $H = \langle A, B\rangle$ then $[A, B]$ is generated by $[\mathfrak{a}_i^\epsilon, \mathfrak{b}_j^\delta]$ where $1 \leq i \leq n$, $1 \leq j \leq m$ and*

$$\epsilon = \begin{cases} 1, & \text{if } |\mathfrak{a}_i| < \infty \\ \pm 1 & \text{if } |\mathfrak{a}_i| = \infty \end{cases} \qquad \delta = \begin{cases} 1, & \text{if } |\mathfrak{b}_j| < \infty \\ \pm 1 & \text{if } |\mathfrak{b}_j| = \infty. \end{cases}$$

PROOF. [Proof of Corollary 21] Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. From the generators and relations of $\nu(G)$ we see that $\nu(G) = \langle G, G^\varphi\rangle$. Hence by Lemma 22 $[G, G^\varphi]$ is generated as needed except for the case $i = j$. It follows from Lemma 10 (i) that $[\mathfrak{g}_i^n, (\mathfrak{g}_i^\varphi)^m] = [\mathfrak{g}_i, \mathfrak{g}_i^\varphi]^{nm}$ for all integers n, m which completes the proof.

In [5], the nonabelian tensor squares for the finite dihedral groups D_n were computed.

Theorem 23 *Let D_m denote the dihedral group of order $2m$. Then*

$$D_m \otimes D_m = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_n, & m \text{ odd;} \\ \mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2, & n \text{ even.} \end{cases}$$

They note that the result holds for the infinite dihedral group denoted by D_0 taking $\mathbb{Z}_0 = \mathbb{Z}$, the infinite cyclic group, but this required a different proof than the one given in the paper for the finite cases.

As an application of our results so far we compute $D_0 \otimes D_0$ where

$$D_0 = \langle a, b \mid a^2, {}^a b = b^{-1}\rangle.$$

By Corollary 21 we have that $D_0 \otimes D_0$ is isomorphic to the subgroup of $\nu(D_0)$ generated by

$$\{[a, a^\varphi], [b, b^\varphi], [a, (b^\varphi)^{\pm 1}], [b^{\pm 1}, a^\varphi]\}.$$

We first show that the generators $[a, (b^\varphi)^{-1}]$ and $[b^{-1}, a^\varphi]$ can be written in terms of the other four generators. We have by Lemma 7 (ii) and the relations of D_0

$$\begin{aligned}
[b^{-1}, a^\varphi] &= {}^{b^{-1}}[b, a^\varphi]^{-1} \\
&= [b^{-1}, [b, a^\varphi]^{-1}][b, a^\varphi]^{-1} \\
&= [b^{-1}, [b, a]^{-\varphi}][b, a^\varphi]^{-1} \\
&= [b^{-1}, (b^\varphi)^{-2}][b, a^\varphi]^{-1} \\
&= [b, b^\varphi]^2[b, a^\varphi]^{-1}.
\end{aligned}$$

Similarly $[a, (b^\varphi)^{-1}]$ can be written in terms of other generators. Hence $[D_0, D_0^\varphi]$ is generated by

$$\{[a, a^\varphi], [b, b^\varphi], [a, b^\varphi], [b, a^\varphi]\}.$$

By Lemma 7 (iii) and (iv), we have that $[a, a^\varphi]$, $[b, b^\varphi]$ and $[a, b^\varphi][b, a^\varphi]$ are all central in $\nu(D_0)$. The product $[a, b^\varphi][b, a^\varphi]$ being a central element implies that $[[a, b^\varphi], [b, a^\varphi] = 1$ since

$$\begin{aligned}
1 &= [[a, b^\varphi][b, a^\varphi], [b, a^\varphi]] \\
&= {}^{[a, b^\varphi]}[[b, a^\varphi], [b, a^\varphi]] \\
&= [[a, b^\varphi], [b, a^\varphi]].
\end{aligned}$$

Hence $G \otimes G$ is abelian.

We will now show that $[a, a^\varphi]$, $[b, b^\varphi]$, $[a, b^\varphi]$ and $[a, b^\varphi][b, a^\varphi]$ have orders 2, 2, 0, and 2 respectively. It follows that $D_0 \otimes D_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_2$ as needed.

Using Lemma 10 (i) we have

$$1 = [a^2, a^\varphi] = [a, a^\varphi]^2.$$

For all the expansions below we use the relations of D_0, $\nu(G)$ and Lemma 7. We first note

$$\begin{aligned}
1 &= [a^2, b^\varphi] = {}^{a}[a, b^\varphi] \cdot [a, b^\varphi] \\
&= [a, b^{-\varphi}] \cdot [a, b^\varphi] && (4) \\
&= [a, [a, b^\varphi]] \cdot [a, b^\varphi]^2. && (5)
\end{aligned}$$

It follows from (4) $[a, b^{-\varphi}] = [a, b^{\varphi}]^{-1}$. Expanding $[a, [a, b^{\varphi}]]$ of (5) we get

$$\begin{aligned} [a, [a, b^{\varphi}]] &= [a, b^{\varphi}, a]^{-1} = [[a, b]^{\varphi}, a]^{-1} \\ &= [(b^{-2})^{\varphi}, a]^{-1} = [b^{-\varphi}, [b^{-\varphi}, a]]^{-1} \cdot [b^{-\varphi}, a]^{-2} \\ &= [b^{-\varphi}, a, b^{-\varphi}] \cdot [b^{-\varphi}, a]^{-2} \\ &= [[b^{-1}, a], b^{-\varphi}] \cdot [b^{-\varphi}, a]^{-2} \\ &= [b^{-2}, b^{-\varphi}] \cdot [b^{-\varphi}, a]^{-2} \\ &= [b, b^{\varphi}]^2 \cdot [a, b^{-\varphi}]^2 \\ &= [b, b^{\varphi}]^2 \cdot [a, b^{\varphi}]^{-2} \end{aligned}$$

Substituting this identity back into (5) we obtain

$$\begin{aligned} 1 = [a^2.b^{\varphi}] &= [a, [a, b^{\varphi}]] \cdot [a, b^{\varphi}]^2 \\ &= [b, b^{\varphi}]^2 \cdot [a, b^{\varphi}]^{-2} \cdot [a, b^{\varphi}]^2 \\ &= [b, b^{\varphi}]^2 \end{aligned}$$

as needed.

We see that

$$\begin{aligned} 1 = [b, (a^2)^{\varphi}] &= [b, a^{\varphi}] \cdot {}^{a^{\varphi}}[b, a^{\varphi}] = [b, a^{\varphi}] \cdot [a^{\varphi}, [b, a^{\varphi}]] \cdot [b, a^{\varphi}] \\ &= [a^{\varphi}, [b, a^{\varphi}]] \cdot [b, a^{\varphi}]^2 \\ &= [b, a^{\varphi}, a^{\varphi}]^{-1} \cdot [b, a^{\varphi}]^2 \\ &= [[b, a]^{\varphi}, a]^{-1} \cdot [b, a^{\varphi}]^2 \\ &= [(b^2)^{\varphi}, a]^{-1} \cdot [b, a^{\varphi}]^2 \\ &= [b^{\varphi}, [b^{\varphi}, a]]^{-1} \cdot [b^{\varphi}, a]^{-2} \cdot [b, a^{\varphi}]^2 \\ &= [b^{\varphi}, a, b^{\varphi}] \cdot [a, b^{\varphi}]^2 \cdot [b, a^{\varphi}]^2 \\ &= [[b, a], b^{\varphi}] \cdot [a, b^{\varphi}]^2 \cdot [b, a^{\varphi}]^2 \\ &= [b^2, b^{\varphi}] \cdot [a, b^{\varphi}]^2 \cdot [b, a^{\varphi}]^2 \\ &= [a, b^{\varphi}]^2 \cdot [b, a^{\varphi}]^2 \\ &= ([a, b^{\varphi}][b, a^{\varphi}])^2 \end{aligned}$$

and the result follows.

We conclude this section by providing some basic structural information for any polycyclic group having a polycyclic generating set of length 2. This coincides with known results on metacyclic groups [5] and [3].

Corollary 24 *Let G be a polycyclic group with a polycyclic generating sequence of length at most 2. Then $G \otimes G$ is abelian and is the direct product of at most 4 cyclic groups.*

PROOF. Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1$, $\mathfrak{g}_2$. Hence ${}^{\mathfrak{g}_1}\mathfrak{g}_2 = \mathfrak{g}_2^{\alpha}$ where α is an integer.

By Corollary 21 we have that $G \otimes G$ is generated by

$$\{[\mathfrak{g}_1, \mathfrak{g}_1^\varphi], [\mathfrak{g}_2, \mathfrak{g}_2^\varphi], [\mathfrak{g}_1^{\pm 1}, (\mathfrak{g}_2^\varphi)^{\pm 1}], [\mathfrak{g}_2^{\pm 1}, (\mathfrak{g}_1^\varphi)^{\pm 1}]\}$$

Now

$$\begin{aligned} [\mathfrak{g}_1^{-1}, \mathfrak{g}_2^\varphi] &= {}^{\mathfrak{g}_1^{-1}}[\mathfrak{g}_1, \mathfrak{g}_2^\varphi]^{-1} \\ &= [{}^{\mathfrak{g}_1^{-1}}\mathfrak{g}_1, ({}^{\mathfrak{g}_1^{-1}}\mathfrak{g}_2)^\varphi]^{-1} \\ &= [\mathfrak{g}_1, (\mathfrak{g}_2^\varphi)^{-\alpha}]. \end{aligned}$$

We now can use the same analysis as in the case of D_0 to show that $G \otimes G$ is abelian and generated by

$$\{[\mathfrak{g}_1, \mathfrak{g}_1^\varphi], [\mathfrak{g}_2, \mathfrak{g}_2^\varphi], [\mathfrak{g}_1, \mathfrak{g}_2^\varphi], [\mathfrak{g}_2, \mathfrak{g}_1^\varphi]\}$$

as needed.

4 Algorithm for computing the tensor square

The goal of this section is to derive from our results in Section 3 a algorithm for computing the nonabelian tensor square for polycyclic groups that can be implemented on a computer.

Whenever, G is polycyclic by Corollary 20 state $\nu(G)$ is polycyclic. Our first task is to find a presentation for $\nu(G)$.

We have already seen that

$$\nu(G) = G * G^\varphi / \langle J \rangle$$

were J is an infinite normal set of words in the free product. We specialize a result of [13] to show that $\langle J \rangle$ is generated by a finite normal set of words which depend on an arbitrary but fixed generating set of G and a polycyclic generating sequence of G.

Theorem 25 *Let G be a polycyclic group. Then the subgroup $\langle J \rangle$ is normally generated by the words*

$$\mathfrak{g}[g_i, g_j^\varphi]\mathfrak{g}^{-1}[{}^{\mathfrak{g}}g_i, ({}^{\mathfrak{g}}g_j)^\varphi]^{-1} \quad \text{and} \quad \mathfrak{g}^\varphi[g_i, g_j^\varphi](\mathfrak{g}^\varphi)^{-1}[{}^{\mathfrak{g}}g_i, ({}^{\mathfrak{g}}g_j)^\varphi]^{-1}$$

where g_i, g_j are elements of $\mathcal{H}$ any generating set for G and $\mathfrak{g}$ is an element of $\mathfrak{G}$ a polycyclic generating sequence for G.

Given a finite presentation of the polycyclic group $G = \langle \mathcal{G} \mid R \rangle$ and polycyclic generating sequence $\mathfrak{G}$ for G, we construct a finite presentation of $\nu(G)$ using

Theorem 25

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^\varphi \mid R, R^\varphi, J \rangle. \tag{6}$$

The problem then is to find a polycyclic presentation for $\nu(G)$ which can be done using a polycyclic quotient algorithm. If this can be done, then the subgroup $[G, G^\varphi] \cong G \otimes G$ can then be computed and its structure examined. Putting these steps together we get the following algorithm.

Algorithm 1 *Given a finite presentation for the polycyclic group $G = \langle \mathcal{G} \mid R \rangle$ with polycyclic generating sequence $\mathfrak{G}$:*

(1) Construct a finite presentation of $\nu(G)$ from $\mathcal{G}$, R and $\mathfrak{G}$.
(2) Compute a polycyclic presentation for $\nu(G)$.
(3) Return a polycyclic subgroup $[G, G^\varphi]$ of $\nu(G)$.

GAP has methods for constructing finitely presented groups as needed in step (1). The Polycyclic GAP package [8] can be used to effectively compute with finite and infinite polycyclic group such as finding subgroup needed in step (3). Hence step (2) is where the real work is done. Two GAP implementations exist to compute the polycyclic quotient of a finitely presented group [12] and [7]. However, neither of them are able to compute a polycyclic presentation of $\nu(G)$ for the simplest infinite example, the infinite dihedral group. Our presentation of $\nu(D_0)$ consists of 4 generators and and 20 relations.

If we turn our attention to finitely generated nilpotent groups the problem of step (2) in the algorithm above reduces to using a nilpotent quotient algorithm since by Theorem **??** $\nu(G)$ is nilpotent. The nq GAP package [14] can effectively find a polycyclic presentation of $\nu(G)$ as needed.

In the next section we will use both the theoretical results of Section 3 and examples to compute the nonabelian tensor square of the free nilpotent groups of class 3 and rank n. A GAP implementation of Algorithm 1 helped in this task in that we abel to compute examples of small rank from which relations for the general case could be determined.

5 Free nilpotent groups of class 3 with finite rank

In this section we apply our results to compute the nonabelian tensor square of the free nilpotent group of class 3 and rank n. We use the theoretical results of Section 3 to aid our work. Computer calculations for $n = 3, 4$ and 5 prompted conjectures that became theoretical results in the analysis given here.

We begin with a series of observations about commutator calculus in $\nu(G)$

when G is a nilpotent group of class at most 3 and in G when G is a nilpotent group of class at most 4.

Note first of all from Theorem 25 that if G is the free nilpotent group of rank n, where $n > 2$, then $\nu(G)$ is nilpotent of class at most 4. If fact $\nu(G)$ must have class exactly 4. Otherwise $\nu(G)'$ would be abelian which implies $G \otimes G$ is abelian, a contradiction since we know that the nonabelian tensor square of the free 2-Engel groups is nilpotent of class 2.

In our context, we are able to extend Lemma 7(i).

Lemma 26 *Suppose the group G is nilpotent of class 3. Then in $\nu(G)$ the identity $[g, h^\varphi, k] = [g^\varphi, h, k]$ holds for all $g, h, k \in G$.*

PROOF. By Lemma 7(iii) $[[g, h^\varphi] \cdot [h, g^\varphi], k] = 1$ in $\nu(G)$ for all $g, h, k \in G$. Since $\nu(G)$ is nilpotent of class at most 4, the product expands linearly, that is, $[g, h^\varphi, k] \cdot [h, g^\varphi, k] = 1$. Hence $[g, h^\varphi, k] = [h, g^\varphi, k]^{-1} = [g^\varphi, h, k]$.

Corollary 27 *Suppose the group G is nilpotent of class 3. Then in $\nu(G)$ the six commutators of Lemma 7(i) are all equal.*

As a consequence, when G is nilpotent of class at most 3, φ may be removed and introduced as is convenient within weight three and weight four commutators in $\nu(G)$ whenever, both in the initial commutator and in the resulting commutator, at least one of the terms is in G and at least one is in G^φ. We use this property in the sequel without further reference. The following case is worth recording.

Lemma 28 *Suppose that G is a nilpotent group of class at most 3. Then for all $g, h, k, l \in G$ the identity*

$$[[g, h], [k, l]^\varphi] = [[g, h^\varphi], [k, l^\varphi]]$$

holds in $\nu(G)$.

The Jacobi identity is used several times in the proofs that follow. For commutators $[x, y] = xyx^{-1}y^{-1}$ and conjugates ${}^y x = yxy^{-1}$, the (general) Jacobi identity involves right-normed commutators:

$${}^y[x, [y^{-1}, z]] \cdot {}^z[y, [z^{-1}, x]] \cdot {}^x[z, [x^{-1}, y]] = 1. \tag{7}$$

We show that the Jacobi identity can be written using left-normed commutators for nilpotent groups of class at most 4.

Lemma 29 *Suppose that G is a nilpotent group of class at most 4, and let $a, b, c \in G$. Then*

$$[a, [b, c]] = [c, b, a].$$

PROOF. We use the identity ${}^{x}[x^{-1}, y]^{-1} = [x, y]$ conjugated by x^{-1}, where $x = [c, b]$ and $y = a$ in the following calculation:

$$\begin{aligned} [a, [b, c]] &= [b, c, a]^{-1} \\ &= [[c, b]^{-1}, a]^{-1} \\ &= {}^{[b,c]}[c, b, a] \\ &= [c, b, a], \end{aligned}$$

where the final equality holds since G is nilpotent of class at most 4.

Applying Lemma 29 to Equation (7) yields the identity

$${}^{y}[z, y^{-1}, x] \cdot {}^{z}[x, z^{-1}, y] \cdot {}^{x}[y, x^{-1}, z] = 1$$

for elements in a nilpotent group of class at most 4. Interchanging x and z throughout gives

$${}^{y}[x, y^{-1}, z] \cdot {}^{x}[z, x^{-1}, y] \cdot {}^{z}[y, z^{-1}, x] = 1,$$

or, since these factors commute in such a group,

$${}^{y}[x, y^{-1}, z] \cdot {}^{z}[y, z^{-1}, x] \cdot {}^{x}[z, x^{-1}, y] = 1. \tag{8}$$

Hence the Jacobi identity holds with left-normed commutators in any group that is nilpotent of class at most 4.

The following result follows immediately.

Lemma 30 *Suppose that G is a nilpotent group of class at most 4, and let $x, y, z, w \in G$. Then*

$$[x, y, z, w] = [x, z, y, w] \cdot [z, y, x, w].$$

PROOF. Since G is nilpotent of class at most 4, computing the commutator of the Jacobi identity (8) with w yields

$$\begin{aligned} 1 = &\ [{}^{y}[x,y^{-1},z]\cdot\ {}^{z}[y,z^{-1},x]\cdot\ {}^{x}[z,x^{-1},y],w] \\ = &\ [\ {}^{y}[x,y^{-1},z],w]\cdot[\ {}^{z}[y,z^{-1},x],w]\cdot[\ {}^{x}[z,x^{-1},y],w] \\ = &\ [x,y^{-1},z,w]\cdot[y,z^{-1},x,w]\cdot[z,x^{-1},y,w] \\ = &\ [x,y,z,w]^{-1}\cdot[y,z,x,w]^{-1}\cdot[z,x,y,w]^{-1}. \end{aligned}$$

Hence $[x,y,z,w]=[y,z,x,w]^{-1}\cdot[z,x,y,w]^{-1}=[z,y,x,w]\cdot[x,z,y,w]$.

For the sake of convenience we list a particular instance of Lemma 30.

Corollary 31 *Suppose that G is a nilpotent group of class at most 3. Then for all $g,h,k,l\in G$ the identity*

$$[g,h,k,l^{\varphi}]=[g,k,h,l^{\varphi}]\cdot[k,h,g,l^{\varphi}]$$

holds in $\nu(G)$.

If G is nilpotent of class at most 3, then $\nu(G)$ is nilpotent of class at most 4 by Theorem 2(iii), and hence any weight four commutator in $\nu(G)$ is central in $\nu(G)$ and weight three commutators commute with weight two commutators. Similarly, inverses can be pulled out directly from weight four commutators and from weight three commutators if the term inverted is a weight two commutator. We use these facts without further comment.

Lemma 32 *Suppose that G is a nilpotent group of class at most 3. Then for all $g,h,k,l\in G$ the identity*

$$[[g,h],[k,l]^{\varphi}]^{-1}=[l,k,h,g^{\varphi}]\cdot[k,l,g,h^{\varphi}]$$

holds in $\nu(G)$.

PROOF. If we let $x=g, y=h^{-1}$ and $z=[k,l]^{\varphi}$ in Equation (8) we get

$$[[g,h],[k,l]^{\varphi}]\cdot[h^{-1},[k,l]^{-\varphi},g]\cdot[[k,l]^{\varphi},g^{-1},h^{-1}]=1.$$

in $\nu(G)$. Hence

$$\begin{aligned} [[g,h],[k,l]^{\varphi}]^{-1} &= [h^{-1},[k,l]^{-\varphi},g]\cdot[[k,l]^{\varphi},g^{-1},h^{-1}] \\ &= [[l,k]^{\varphi},h,g]\cdot[[k,l]^{\varphi},g,h] \\ &= [l,k,h,g^{\varphi}]\cdot[k,l,g,h^{\varphi}] \end{aligned}$$

The identities of Corollary 31 and Lemma 32 are used in the proof of Lemma 33.

Lemma 33 *Suppose that G is a nilpotent group of class at most 3. Then for all $g, h, k \in G$ the identity*

$$[g,h,k^{\varphi}] = [k,h,g^{\varphi}]\cdot[g,k,h^{\varphi}]\cdot[[g,h],[g,k]^{\varphi}]^{-1}\cdot[[g,h],[h,k]^{\varphi}]^{-1}\cdot[[g,k],[h,k]^{\varphi}]^{-1} \tag{9}$$

holds in $\nu(G)$.

PROOF. The choices $x = g, y = h^{-1}$ and $z = k^{\varphi}$ in the Jacobi identity (8) yield the equation

$$^{h^{-1}}[g,h,k^{\varphi}]\cdot\ ^{k^{\varphi}}[h^{-1},k^{-\varphi},g]\cdot\ ^{g}[k^{\varphi},g^{-1},h^{-1}] = 1.$$

By the identity $^{x}y = [x,y]y$ we obtain

$$\begin{aligned}[h^{-1},[g,h,k^{\varphi}]]\cdot[g,h,k^{\varphi}]\cdot[k^{\varphi},[h^{-1},k^{-\varphi},g]]\cdot[h^{-1},k^{-\varphi},g]\\ [g,[k^{\varphi},g^{-1},h^{-1}]]\cdot[k^{\varphi},g^{-1},h^{-1}] = 1.\end{aligned}$$

We simplify each factor of this product (except the second) one at a time:

$[h^{-1},[g,h,k^{\varphi}]] = [g,h,k^{\varphi},h] = [g,h,k,h^{\varphi}];$

$[k^{\varphi},[h^{-1},k^{-\varphi},g]] = [h,k^{\varphi},g,k^{\varphi}]^{-1} = [h,k,g,k^{\varphi}]^{-1};$

$$\begin{aligned}[h^{-1},k^{-\varphi},g] &= [\ ^{h^{-1}}[h,k^{-\varphi}]^{-1},g]\\ &= [[h^{-1},[h,k^{-\varphi}]^{-1}]\cdot[h,k^{-\varphi}]^{-1},g]\\ &= [[h^{-1},[h,k^{-\varphi}]^{-1}],g]\cdot[[h,k^{-\varphi}]^{-1},g]\\ &= [h,k^{-\varphi},h,g]^{-1}\cdot[h,k^{-\varphi},g]^{-1}\\ &= [h,k^{\varphi},h,g]\cdot[\ ^{k^{-\varphi}}[h,k^{\varphi}]^{-1},g]^{-1}\\ &= [h,k,h,g^{\varphi}]\cdot[[k^{-\varphi},[h,k^{\varphi}]^{-1}\cdot[h,k^{\varphi}]^{-1},g]^{-1}\\ &= [h,k,h,g^{\varphi}]\cdot[k^{-\varphi},[h,k^{\varphi}]^{-1},g]^{-1}\cdot[[h,k^{\varphi}]^{-1},g]^{-1}\\ &= [h,k,h,g^{\varphi}]\cdot[h,k^{\varphi},k^{-\varphi},g]^{-1}\cdot[h,k^{\varphi},g]\\ &= [h,k,h,g^{\varphi}]\cdot[h,k,k,g^{\varphi}]\cdot[h,k,g^{\varphi}];\end{aligned}$$

$[g,[k^{\varphi},g^{-1},h^{-1}]] = [[k^{\varphi},g^{-1},h^{-1}],g]^{-1} = [k,g,h,g^{\varphi}]^{-1};$

and

$$\begin{aligned}
[k^\varphi, g^{-1}, h^{-1}] &= [\ {}^{g^{-1}}[k^\varphi, g]^{-1}, h^{-1}] \\
&= [[g^{-1}, [k^\varphi, g]^{-1}] \cdot [k^\varphi, g]^{-1}, h^{-1}] \\
&= [[g^{-1}, [k^\varphi, g]^{-1}], h^{-1}] \cdot [[k^\varphi, g]^{-1}, h^{-1}] \\
&= [k^\varphi, g, g, h] \cdot [[k^\varphi, g], h^{-1}]^{-1} \\
&= [k, g, g, h^\varphi] \cdot {}^{h^{-1}}[[k^\varphi, g], h] \\
&= [k, g, g, h^\varphi] \cdot [h^{-1}, [[k^\varphi, g], h]] \cdot [[k^\varphi, g], h] \\
&= [k, g, g, h^\varphi] \cdot [k^\varphi, g, h, h] \cdot [k, g, h^\varphi] \\
&= [k, g, g, h^\varphi] \cdot [k, g, h, h^\varphi] \cdot [k, g, h^\varphi].
\end{aligned}$$

Hence we have

$$\begin{aligned}
[g, h, k^\varphi] = {} & [h, g, k, h^\varphi] \cdot [h, k, g, k^\varphi] \cdot [k, h, h, g^\varphi] \cdot [k, h, k, g^\varphi] \cdot [k, h, g^\varphi] \cdot \\
& [k, g, h, g^\varphi] \cdot [g, k, g, h^\varphi] \cdot [g, k, h, h^\varphi] \cdot [g, k, h^\varphi].
\end{aligned}$$

By Corollary 31 we replace $[h, g, k, h^\varphi] \cdot [g, k, h, h^\varphi]$ with $[h, k, g, h^\varphi]$, which yields

$$\begin{aligned}
[g, h, k^\varphi] = {} & [h, k, g, k^\varphi] \cdot [k, h, h, g^\varphi] \cdot [k, h, k, g^\varphi] \cdot [k, h, g^\varphi] \cdot \\
& [k, g, h, g^\varphi] \cdot [g, k, g, h^\varphi] \cdot [h, k, g, h^\varphi] \cdot [g, k, h^\varphi]. \qquad (10)
\end{aligned}$$

By Lemma 32 with suitable substitutions we obtain

$$[k, g, h, g^\varphi] \cdot [g, k, g, h^\varphi] = [[g, h], [g, k]^\varphi]^{-1},$$

$$[k, h, h, g^\varphi] \cdot [h, k, g, h^\varphi] = [[g, h], [h, k]^\varphi]^{-1}$$

and

$$[k, h, k, g^\varphi] \cdot [h, k, g, k^\varphi] = [[g, k], [h, k]^\varphi]^{-1}.$$

Making the appropriate substitutions in Equation (10) yields

$$\begin{aligned}
[g, h, k^\varphi] = {} & [k, h, g^\varphi] \cdot [g, k, h^\varphi] \cdot \\
& [[g, h], [g, k]^\varphi]^{-1} \cdot [[g, h], [h, k]^\varphi]^{-1} \cdot [[g, k], [h, k]^\varphi]^{-1}.
\end{aligned}$$

Corollary 34 *Suppose that G is a nilpotent group of class at most 3. The following identities hold in $\nu(G)$:*

(i) $[g, h, k, l^\varphi] = [[g, h], [k, l]^\varphi] \cdot [g, h, l, k^\varphi]$ *for all* $g, h, k, l \in G$*;*
(ii) $[g, h, h, k^\varphi] = [[g, h], [h, k]^\varphi] \cdot [g, h, k, h^\varphi]$ *for all* $g, h, k \in G$*;*
(iii) $[g, h, h, g^\varphi] = [g, h, g, h^\varphi]$ *for all* $g, h \in G$.

PROOF. (i) If we replace g by $[g, h]$, h by k and k by l in Identity (9), and note that commutators of weight 5 are trivial in $\nu(G)$, we obtain

$$\begin{aligned}[][g,h],k,l^{\varphi}] &= [l,k,[g,h]^{\varphi}] \cdot [[g,h],l,k^{\varphi}] \\ &= [[l,k]^{\varphi},[g,h]] \cdot [g,h,l,k^{\varphi}] \\ &= [[[g,h],[k,l]^{\varphi}] \cdot [g,h,l,k^{\varphi}].\end{aligned}$$

(ii) and (iii) are special cases of (i).

Lemma 35 *Suppose that G is a nilpotent group of class at most 3. Then for all $g,h,k,l \in G$ the identity*

$$[g,h,k,l^{\varphi}] = [g,h,l,k^{\varphi}] \cdot [k,l,g,h^{\varphi}]^{-1} \cdot [k,l,h,g^{\varphi}]$$

holds in $\nu(G)$.

PROOF. By Lemma 33 with g replaced by $[g,h]$, h replaced by k and k replaced by l, we have

$$[g,h,k,l^{\varphi}] = [[l,k],[g,h]^{\varphi}] \cdot [g,h,l,k^{\varphi}] = [[g,h],[k,l]^{\varphi}] \cdot [g,h,l,k^{\varphi}]. \qquad (11)$$

Also by Lemma 33, this time with g replaced by $[k,l]$, h replaced by g and k replaced by h, we have

$$[k,l,g,h^{\varphi}] = [[h,g],[k,l]^{\varphi}] \cdot [k,l,h,g^{\varphi}] = [[g,h],[k,l]^{\varphi}]^{-1} \cdot [k,l,h,g^{\varphi}],$$

and hence

$$[[g,h],[k,l]^{\varphi}] = [k,l,g,h^{\varphi}]^{-1} \cdot [k,l,h,g^{\varphi}].$$

Substitution into Equation (11) yields the result.

We conclude our results on the commutator calculus of $\nu(G)$ when G is nilpotent of class at most 3 with the following lemma.

Lemma 36 *Suppose that G is a nilpotent group of class at most 3. Then for all $g,h,k \in G$ the following identities hold in $\nu(G)$:*

(i) $[[x,y]^{-1},z^{\varphi}] = [[x,y],z^{\varphi}]^{-1}$;

(ii) $[[x,y],z^{-\varphi}] = [[x,y,z],z^{\varphi}]^{-1} \cdot [[x,y],z^{\varphi}]^{-1}$;

(iii) $[[x,y]^{-1},z^{-\varphi}] = [[x,y,z],z^{\varphi}] \cdot [[x,y],z^{\varphi}]$;

(iv) $[x^{-1},y^{\varphi}] = [[x,y,x],x^{\varphi}]^{-1} \cdot [[x,y],x^{\varphi}]^{-1} \cdot [x,y^{\varphi}]^{-1}$;

(v) $[x,y^{-\varphi}] = [[x,y,y],y^{\varphi}]^{-1} \cdot [[x,y],y^{\varphi}]^{-1} \cdot [x,y^{\varphi}]^{-1}$; *and*

(vi) $[x^{-1}, y^{-\varphi}] = [[x, y, x], x^\varphi] \cdot [[x, y, y], x^\varphi][[x, y, y], y^\varphi][[x, y], x^\varphi] \cdot [[x, y], y^\varphi] \cdot [x, y^\varphi]$.

I can't figure out how to get the overflow line of (vi) to move right past the = sign of the previous line. TeX-smith?

PROOF. Although (i)-(iii) are special cases of (iv)-(vi), we use the former to prove the latter. From the fact that $\nu(G)$ is nilpotent of class at most 4, we have $[[x, y]^{-1}, z^\varphi] = {}^{[x,y]^{-1}}[[x, y], z^\varphi]^{-1} = [[x, y], z^\varphi]^{-1}$, that is, (i) holds. For the proof of (ii) we use Corollary 27:

$$\begin{aligned}
[[x, y], z^{-\varphi}] &= {}^{z^{-\varphi}}[[x, y], z^\varphi]^{-1} \\
&= [z^{-\varphi}, [[x, y], z^\varphi]^{-1}] \cdot [[x, y], z^\varphi]^{-1} \\
&= [z^\varphi, [[x, y], z^\varphi]] \cdot [[x, y], z^\varphi]^{-1} \\
&= [[[x, y], z^\varphi], z^\varphi]^{-1} \cdot [[x, y], z^\varphi]^{-1} \\
&= [[x, y, z], z^\varphi]^{-1} \cdot [[x, y], z^\varphi]^{-1},
\end{aligned}$$

which proves (ii). By (i) and (ii), $[[x, y]^{-1}, z^{-\varphi}] = [[x, y], z^{-\varphi}]^{-1} = [[x, y, z], z^\varphi] \cdot [[x, y], z^\varphi]$, that is, (iii) holds. Next, using (ii),

$$\begin{aligned}
[x^{-1}, y^\varphi] &= {}^{x^{-1}}[x, y^\varphi]^{-1} \\
&= [x^{-1}, [x, y^\varphi]^{-1}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y^\varphi]^{-1}, x^{-1}]^{-1} \cdot [x, y^\varphi]^{-1} \\
&= {}^{[x,y^\varphi]^{-1}}[[x, y^\varphi], x^{-1}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y^\varphi], x^{-1}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y], x^{-\varphi}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y, x], x^\varphi]^{-1} \cdot [[x, y], x^\varphi]^{-1} \cdot [x, y^\varphi]^{-1},
\end{aligned}$$

which proves (iv). We also use (ii) to prove (v):

$$\begin{aligned}
[x, y^{-\varphi}] &= {}^{y^{-\varphi}}[x, y^\varphi]^{-1} \\
&= [y^{-\varphi}, [x, y^\varphi]^{-1}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y^\varphi]^{-1}, y^{-\varphi}]^{-1} \cdot [x, y^\varphi]^{-1} \\
&= {}^{[x,y^\varphi]^{-1}}[[x, y^\varphi], y^{-\varphi}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y^\varphi], y^{-\varphi}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y], y^{-\varphi}] \cdot [x, y^\varphi]^{-1} \\
&= [[x, y, y], y^\varphi]^{-1} \cdot [[x, y], y^\varphi]^{-1} \cdot [x, y^\varphi]^{-1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
[x^{-1}, y^{-\varphi}] &= [[x, y^{-1}, x], x^{\varphi}]^{-1} \cdot [[x, y^{-1}], x^{\varphi}]^{-1} \cdot [x, y^{-\varphi}]^{-1} \\
&= [[x, y, x], x^{\varphi}] \cdot [[x, y^{-\varphi}], x^{\varphi}]^{-1} \cdot [x, y^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [[x, y, y], y^{\varphi}] \\
&= [[x, y, x], x^{\varphi}] \cdot [[[x, y, y], y^{\varphi}]^{-1} \cdot [[x, y], y^{\varphi}]^{-1} \cdot [x, y^{\varphi}]^{-1}, x^{\varphi}]^{-1} \cdot \\
&\quad\; [x, y^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [[x, y, y], y^{\varphi}] \\
&= [[x, y, x], x^{\varphi}] \cdot [[[x, y], y^{\varphi}]^{-1} \cdot [x, y^{\varphi}]^{-1}, x^{\varphi}]^{-1} \cdot \\
&\quad\; [x, y^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [[x, y, y], y^{\varphi}] \\
&= [[x, y, x], x^{\varphi}] \cdot ({}^{[[x,y],y^{\varphi}]^{-1}}[[x, y^{\varphi}]^{-1}, x^{\varphi}] \cdot [[[x, y], y^{\varphi}]^{-1}, x^{\varphi}])^{-1} \cdot \\
&\quad\; [x, y^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [[x, y, y], y^{\varphi}] \\
&= [[x, y, x], x^{\varphi}] \cdot ({}^{[x,y^{\varphi}]^{-1}}[[x, y^{\varphi}], x^{\varphi}]^{-1} \cdot {}^{[[x,y],y^{\varphi}]^{-1}}[[[x, y], y^{\varphi}], x^{\varphi}]^{-1})^{-1} \\
&\quad\; [x, y^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [[x, y, y], y^{\varphi}] \\
&= [[x, y, x], x^{\varphi}] \cdot [[[x, y], y^{\varphi}], x^{\varphi}][[x, y^{\varphi}], x^{\varphi}] \cdot \\
&\quad\; [x, y^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [[x, y, y], y^{\varphi}] \\
&= [[x, y, x], x^{\varphi}] \cdot [[x, y, y], x^{\varphi}][[x, y], x^{\varphi}] \cdot \\
&\quad\; [x, y^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [[x, y, y], y^{\varphi}] \\
&= [[x, y, x], x^{\varphi}] \cdot [[x, y, y], x^{\varphi}][[x, y, y], y^{\varphi}][[x, y], x^{\varphi}] \cdot [[x, y], y^{\varphi}] \cdot [x, y^{\varphi}],
\end{aligned}$$

which proves (vi).

We now have the machinery available to describe the nonabelian tensor square of the free nilpotent group of class 3 and rank n. Let G be the free nilpotent of class 3 group of rank n generated by $g_1, \ldots, g_n$. Every element g of G can be written in the form

$$g = \prod_{1 \le i \le n} g_i^{\alpha_i} \prod_{1 \le i < j \le n} [g_i, g_j]^{\beta_{i,j}} \prod_{1 \le i < j < k \le n} [g_i, g_j, g_k]^{\rho_{i,j,k}} [g_j, g_k, g_i]^{\sigma_{i,j,k}} \prod_{1 \le i,j \le n, i \ne j} [g_i, g_j, g_j]^{\gamma_{i,j}},$$

where the $\alpha_i, \beta_{i,j}, \rho_{i,j,k}, \sigma_{i,j,k}$ and $\gamma_{i,j}$ are integers, and we have a polycyclic generating set

$$\begin{aligned}
\{g_i \mid 1 \le i \le n\} &\cup \{[g_i, g_j] \mid 1 \le i < j \le n\} \\
&\cup \{[g_i, g_j, g_k], [g_j, g_k, g_i] \mid 1 \le i < j < k \le n\} \\
&\cup \{[g_i, g_j, g_j] \mid 1 \le i, j \le n, i \ne j\}.
\end{aligned} \tag{12}$$

for G.

Recall from Theorem 21 that the subgroup $[G, G^{\varphi}] \cong G \otimes G$ of $\nu(G)$ is generated by commutators $[\mathfrak{g}^{\epsilon}, (\mathfrak{h}^{\varphi})^{\delta}]$, where $\mathfrak{g}$ and $\mathfrak{h}$ are elements of a polycyclic generating sequence for G and $\epsilon, \delta \in \{-1, 1\}$. By Lemma 36 we need only retain commutators of the form $[\mathfrak{g}, \mathfrak{h}^\varphi]$, where $\mathfrak{g}$ and $\mathfrak{h}$ are elements of a polycyclic generating sequence for G. Thus we consider which among the possible such commutators of elements (12) and their images under φ are needed to generate $[G, G^\varphi]$. Certainly only resulting commutators of weight at most 4 need be considered; also by Corollary 27 we need consider only commutators in which the weight of $\mathfrak{g}$ is at least the weight of $\mathfrak{h}^\varphi$.

From the nilpotency restriction on $\nu(G)$ and Lemma 7(iii) we see that the only generators of $[G, G^\varphi]$ not in the center of $[G, G^\varphi]$ are of the form

$$[g_i, g_r^\varphi], \qquad 1 \le i, r \le n, \quad i \ne r. \tag{13}$$

There are

$$n(n-1) \tag{14}$$

such generators of $[G, G^\varphi]$. We denote by N the subgroup of $[G, G^\varphi]$ generated by the commutators of the form (13); this subgroup is nilpotent of class 2. Generators of $[G, G^\varphi]$ of the form

$$[[g_i, g_j], [g_r, g_s]^\varphi] = [[g_i, g_j^\varphi], [g_r, g_s^\varphi]]$$

lie in (the center of) N, and thus are not needed to generate $[G, G^\varphi]$.

How to phrase? Keep the reference to B throughout, or drop back to A as quickly as possible

Since all other generators of $[G, G^\varphi]$ are central in $[G, G^\varphi]$, the subgroup B generated by them can be factored as a direct product $(B \cap N) \times A$ for some suitable subgroup A of B; note that A is unique up to abelian invariants. In what follows we describe and count a minimal set of generators for such a subgroup A. We start by noting that here are

$$n \tag{15}$$

generators of B of form $[g_i, g_i^\varphi]$ for $1 \le i \le n$, all of which we retain as generators of A.

Consider next generators of B of form $[g_i, g_j, g_r^\varphi]$ for $1 \le i < j \le n, 1 \le r \le n$. By Lemma 33 one of the generators $[g_i, g_j, g_r^\varphi], [g_r, g_j, g_i^\varphi]$ (or its inverse $[g_j, g_r, g_i^\varphi]$, if $j < r$) or $[g_i, g_r, g_j^\varphi]$ (or its inverse $[g_r, g_i, g_j^\varphi]$, if $r < i$) is not needed, since one of these generators is the product of the other two generators and an element of $B \cap N$. Thus for each possible selection of i, j and r, we retain 2 generators. There are $\binom{n}{3}$ ways to choose i, j and r if $r \notin \{i, j\}$ and $\binom{n}{2}$ ways to choose i, j and r if $r \in \{i, j\}$. Hence we retain as generators of A

$$2\binom{n}{3} + 2\binom{n}{2} \tag{16}$$

generators of the form $[g_i, g_j, g_r^\varphi]$.

For weight 4 generators of B of the form $[g_i, g_j, g_k, g_r^\varphi]$ and $[g_j, g_k, g_i, g_r^\varphi]$ for $1 \le i < j < k \le n$ and $1 \le r \le n$ we retain all

$$6\binom{n}{3} \tag{17}$$

generators arising from the possible choices of $i < j < k$ and $r \in \{i, j, k\}$. For the generators arising when i, j, k and r are distinct, we consider at one time all such generators arising for a particular selection of indices $1 \le t < u < v < w \le n$. Four separate pairs of generators appear depending on which of t, u, v or w corresponds to r; the other indices then correspond in one way to $i < j < k$. The generators that arise are thus:

$$\begin{aligned}&[g_u, g_v, g_w, g_t^\varphi], [g_v, g_w, g_u, g_t^\varphi], [g_t, g_v, g_w, g_u^\varphi], [g_v, g_w, g_t, g_u^\varphi],\\ &[g_t, g_u, g_w, g_v^\varphi], [g_u, g_w, g_t, g_v^\varphi], [g_t, g_u, g_v, g_w^\varphi], [g_u, g_v, g_t, g_w^\varphi].\end{aligned} \tag{18}$$

(The other 16 of the possible $4! = 24$ weight four commutators are expressible in terms of these 8 by inversion and the Jacobi identity.) By Corollary 34(i) each pair of commutators

$$\begin{gathered}\{[g_u, g_v, g_w, g_t^\varphi], [g_u, g_v, g_t, g_w^\varphi]\}\\ \{[g_v, g_w, g_u, g_t^\varphi], [g_v, g_w, g_t, g_u^\varphi]\}\\ \{[g_t, g_u, g_w, g_v^\varphi], [g_t, g_u, g_v, g_w^\varphi]\}\end{gathered}$$

from the list (18) differs by an element of $B \cap N$, and thus one of each pair can be removed as a generator. We may therefore reduce the list of generators of this type to the following:

$$[g_u, g_v, g_w, g_t^\varphi], [g_v, g_w, g_u, g_t^\varphi], [g_t, g_v, g_w, g_u^\varphi], [g_t, g_u, g_w, g_v^\varphi], [g_u, g_w, g_t, g_v^\varphi]. \tag{19}$$

Lemma 35 tells us that

$$\begin{aligned}[g_t, g_u, g_w, g_v^\varphi] &= [g_t, g_u, g_v, g_w^\varphi] \cdot [g_w, g_v, g_t, g_u^\varphi]^{-1} \cdot [g_w, g_v, g_u, g_t^\varphi]\\ &= [g_t, g_u, g_v, g_w^\varphi] \cdot [g_v, g_w, g_t, g_u^\varphi] \cdot [g_v, g_w, g_u, g_t^\varphi]^{-1} \qquad (20)\\ &\qquad (21)\end{aligned}$$

and

$$\begin{aligned}[g_t, g_v, g_w, g_u^\varphi] &= [g_t, g_v, g_u, g_w^\varphi] \cdot [g_w, g_u, g_t, g_v^\varphi]^{-1} \cdot [g_w, g_u, g_v, g_t^\varphi]\\ &= [g_v, g_t, g_u, g_w^\varphi]^{-1} \cdot [g_u, g_w, g_t, g_v^\varphi] \cdot [g_w, g_u, g_v, g_t^\varphi]. \qquad (22)\\ &\qquad (23)\end{aligned}$$

We may thus remove from (19) one of the two commutators that appears in Equation 20 and likewise for Equation 22; we arrive then at a list of 3 commutators that we retain as generators of A for each selection $1 \le t < u < v < w \le n$, for example,

$$[g_u, g_v, g_w, g_t^\varphi], [g_t, g_v, g_w, g_u^\varphi], [g_t, g_u, g_w, g_v^\varphi]. \tag{24}$$

We conclude that we retain

$$3\binom{n}{4} \tag{25}$$

generators of form $[g_i, g_j, g_k, g_r^\varphi]$ and $[g_j, g_k, g_i, g_r^\varphi]$ for $1 \le i < j < k \le n$ and $1 \le r \le n$ and $r \notin \{i, j, k\}$.

We consider finally generators of B of form $[g_i, g_j, g_j, g_r^\varphi]$ for $1 \le i, j \le n$ and $1 \le r \le n$. We again consider two possibilities. If $r \in \{i, j\}$ we have four potential generators of this form for each choice of the subset $\{i, j\}$, namely $[g_i, g_j, g_j, g_i^\varphi], [g_i, g_j, g_j, g_j^\varphi], [g_j, g_i, g_i, g_i^\varphi]$ and $[g_j, g_i, g_i, g_j^\varphi]$. By Corollary 34(iii) the two commutators $[g_i, g_j, g_j, g_i^\varphi]$ and $[g_i, g_j, g_i, g_j^\varphi]$ differ by an element of $B \cap N$. We therefore retain

$$3\binom{n}{2} \tag{26}$$

generators of the form $[g_i, g_j, g_j, g_r^\varphi]$ when $r \in \{i, j\}$. If $r \notin \{i, j\}$, then Corollary 34(ii) shows that $[g_i, g_j, g_j, g_r^\varphi]$ differs from $[g_i, g_j, g_r, g_j^\varphi]$ by an element of $B \cap N$, and since we have already accounted for all generators of form $[g_i, g_j, g_r, g_j^\varphi]$ when i, j and r are distinct, we retain no generators of form $[g_i, g_j, g_j, g_r^\varphi]$ when $r \notin \{i, j\}$.

In this revision I am not yet tackling the question of proving isomorphism. I wanted to get at least the part I have tackled written and looked at, and I'm not quite sure how to present the isomorphism argument yet. I will likely need to talk with you about this. Also, the theorem here is likely to be placed in Section 1, so that the following gets replaced by something like "Proof of Theorem xx". Question: presumably N should be free nilpotent of class 2, not just nilpotent of class 2.

Theorem 37 *Let G be a free nilpotent group of class 3 and rank n. Then $G \otimes G \cong N \times A$ where N is nilpotent of class 2 with rank $n(n-1)$ and A is free abelian of rank $f(n) = \frac{n(3n^3+14n^2-3n+10)}{24}$.*

PROOF. The only remaining observation to make is that by (15),(16),(17),(25) and (26) the rank $f(n)$ of A is

$$\begin{aligned} f(n) &= n + 2\binom{n}{3} + 2\binom{n}{2} + 6\binom{n}{3} + 3\binom{n}{4} + 3\binom{n}{2} \\ &= n + 5\binom{n}{2} + 8\binom{n}{3} + 3\binom{n}{4} \\ &= \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}. \end{aligned}$$

The ranks of N and A have been checked computationally by direct calculations in $[G, G^\varphi]$ using GAP when the rank n of the free nilpotent group G of class 3 is $n = 3, 4, 5$ and 6.

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