

Computing the nonabelian tensor square group

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Organizers

Arturo Magidin, University of Louisiana at Lafayette
Luise Charlotte Kappe, Binghamton University
Robert F. Morse, University of Evansville

The main objective of the special session will be to feature group theoretic results obtained through the application of Computational Group Theory.

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Definition

Given a group G we define the nonabelian tensor square to be the group generated by the symbols

$$g \otimes g' \quad \text{for all } g, g' \in G$$

subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h)$$

$$g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$$

where ${}^x y = xyx^{-1}$.

Background

The nonabelian tensor square is a specialization of the more general nonabelian tensor product introduced by Ronald Brown and Jean-Louis Loday (1987).

This group construction which has its roots in algebraic topology and topology.

Ideas of this construction can be found in Whitehead's work from the 1950s.

The nonabelian tensor square appears independently in Dennis's work (1976) in K -theory and is based on ideas from Whitehead (1952).

Background (cont)

The investigation of this group construction from a group-theoretic point of view started with a paper by Ronald Brown, David Johnson, and Edmund Robertson (1987).

The goals of their investigation include:

- Compute the nonabelian tensor square for a given group G and give a description of $G \otimes G$ that is simplified and easy to recognize.
- Determine the structure of $G \otimes G$ from the structure of G .
- Compute homomorphisms of $G \otimes G$.

Commutative Diagram BJR (1987)

Exact rows and central extensions as columns.

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) \\
 \parallel & & \parallel & & \downarrow & & \downarrow \\
 H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \xrightarrow{\psi} & G \otimes G & \longrightarrow & G \wedge G \\
 & & & & \kappa \downarrow & & \kappa' \downarrow \\
 & & & & G' & \xlongequal{\quad} & G' \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}$$

Whitehead's Quadratic Functor (ΓA)

Let A be an abelian group. The *Whitehead Quadratic* functor ΓA is the abelian group with generators γa , $a \in A$, and defining

$$\begin{aligned}\gamma(a^{-1}) &= \gamma a \\ \gamma(abc)\gamma a\gamma b\gamma c &= \gamma(ab)\gamma(bc)\gamma(ca)\end{aligned}$$

For finitely generated abelian groups ΓA can be directly

$$(a) \quad \Gamma(A \times B) \cong \Gamma A \times \Gamma B \times (A \otimes B)$$

$$(b) \quad \Gamma Z_n \cong \begin{cases} Z_n, & n \text{ odd} \\ Z_{2n}, & n \text{ even} \end{cases}$$

Other related groups

The groups $H_2(G)$ and $H_3(G)$ are the second and third homology groups. $H_2(G)$ is also known as the Schur Multiplier.

The group $J_2(G)$ is isomorphic to the third homotopy group of the suspension of an Eilenberg-MacLane space.

The group $G \wedge G$ is the nonabelian exterior square of G . It is isomorphic to the quotient group $(G \otimes G) / \nabla(G)$ where $\nabla(G) = \langle x \otimes x \mid x \in G \rangle$ is a subgroup of $G \otimes G$.

Structure Results

If G is finite then $G \otimes G$ is finite.

If G' is nilpotent of class c then $G \otimes G$ is nilpotent of class $c + 1$ (MB, LCK, RFM).

If G' is solvable of derived length l then $G \otimes G$ is solvable of derived length at most $l + 1$ (MV)

If G is polycyclic then $G \otimes G$ is polycyclic (RFM).

Computing the tensor square

We have only the left conjugation action of G to work with, which dictates what $G \otimes G$ is going to be.

For a finite group G , the definition gives us a finite presentation of $G \otimes G$. We can apply Tietze transformations to this presentation to obtain a simplified presentation of $G \otimes G$. We can then use this simplified presentation to determine (in a more standard way) what the tensor square is.

Brown, Johnson, and Robertson (1987) compute the nonabelian tensor square of all nonabelian groups up to order 30 using Tietze transformations. This method does not scale well as we have $|G|$ generators and $2|G|^3$ relations.

The group $\nu(G)$

Rocco (1991) considers the following group. Let G and G^φ be isomorphic groups via $\varphi : g \mapsto g^\varphi$, for all $g \in G$. Then

$$\nu(G) = \langle G, G^\varphi \mid {}^k[g, h^\varphi] = [{}^k g, ({}^k h)^\varphi] = {}^{k^\varphi}[g, h^\varphi], \forall g, h \in G \rangle$$

Rocco investigates the structural aspects of $\nu(G)$ relative to G .

Theorem 1. *Let G be a group.*

- (i) *If G is finite then $\nu(G)$ is finite.*
- (ii) *If G is a finite p -group then $\nu(G)$ is a finite p -group.*
- (iii) *If G is nilpotent of class c then $\nu(G)$ is nilpotent of class at most $c + 1$.*
- (iv) *If G is solvable of derived length d then $\nu(G)$ is solvable of derived length at most $d + 1$.*

The group $\nu(G)$ (cont)

The group $\nu(G)$ actually has its roots in a more general crossed modules noted by Ellis in Ellis and Leonard (1995).

The importance of this construction is the following:

Theorem 2. *Let G be a group. The map*

$\sigma : G \otimes G \rightarrow [G, G^\varphi] \triangleleft \nu(G)$ *defined by $\sigma(g \otimes h) = [g, h^\varphi]$ is an isomorphism.*

Ellis and Leonard (1995) give one more structure result.

Theorem 3. *Let G be a group. Then*

$$1 \longrightarrow [G, G^\varphi] \longrightarrow \nu(G) \longrightarrow G \times G \longrightarrow$$

is a short exact sequence.

The group $\nu(G)$ (cont)

Definition 4. Let G be a group and let

$G = G_n \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq G_0 = 1$ be a subnormal series for G . For each i , denote a transversal for G_{i-1} in G_i and let \mathcal{G}_i denote a generating set for G_i/G_{i-1} to \mathcal{T}_i . Set

$$\mathcal{L}_i = \begin{cases} \mathcal{G}_i, & \text{if } G_i/G_{i-1} \text{ is abelian} \\ \mathcal{T}_i, & \text{otherwise.} \end{cases}$$

Then define the set \mathcal{L}_G to be

$$\mathcal{L}_G = \cup_{i=1}^n \mathcal{L}_i.$$

The group $\nu(G)$ (cont)

The following theorem is by McDermott (1998) which extends the work of Ellis:

Theorem 5. *Let G be a group generated by a set \mathcal{G} . Then $\nu(G) = G * G^\varphi / \langle J \rangle$, where J is the normal generating set of the elements*

$${}^x[a, b^\varphi][{}^x a, ({}^x b)^\varphi]^{-1}, {}^{x^\varphi}[a, b^\varphi][{}^x a, ({}^x b)^\varphi]^{-1}$$

for all a, b in \mathcal{G} and x in \mathcal{L}_G .

Let $G = \langle \mathcal{G}, \mathcal{R} \rangle$. The theorem shows the defining presentation for $\nu(G)$ where can be reduced to $2|\mathcal{G}|$ generators and $2|\mathcal{R}| + 2|\mathcal{G}|^2 \cdot |\mathcal{L}_G|$.

This is a reasonably small presentation to compute and work with.

Two algorithms

For both algorithms we

- 1 Compute a finite presentation for $\nu(G)$ from some \mathcal{L}
finite group we can choose $G \triangleright Z(G) \triangleright 1$ to form \mathcal{L}

Algorithm One:

- 2a Find a concrete representation for $\nu(G)$ (p -quotient,
nilpotent quotient, solvable quotient, coset enumeration)
- 3a Compute $[G, G^\varphi]$.

Algorithm Two:

- 2b Compute the kernel of the mapping $\nu(G) \rightarrow G \times G$.

Two algorithms (cont)

Polycyclic groups are those groups with a finite subnormal series such that each factor is cyclic. These groups are the infinite analogue of the “power commutator” (pc) groups.

All polycyclic groups are solvable and have a well developed theory such that we can compute effectively with them nearly as well as with finite groups.

Let G be a polycyclic group with generating set \mathcal{G} and presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$. Then we can set $\mathcal{L}_G = \mathcal{G}$. Moreover

Theorem 6. *Suppose that G is a polycyclic group. Then G is finitely presented and $G \otimes G$ are also polycyclic.*

Two algorithms (cont)

Corollary 7. *Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \dots, \mathfrak{g}_k$. Then the subgroup $[G, G]$ is generated by $\{[\mathfrak{g}_i, \mathfrak{g}_i^\varphi], [\mathfrak{g}_i^\epsilon, (\mathfrak{g}_j^\varphi)^\delta]\}$ for $1 \leq i, j, \leq k, i \neq j$.*

$$\epsilon = \begin{cases} 1, & \text{if } |\mathfrak{g}_i| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_i| = \infty \end{cases} \quad \delta = \begin{cases} 1, & \text{if } |\mathfrak{g}_j^\varphi| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_j^\varphi| = \infty \end{cases}$$

A simple example

Let $G = GL(2, 3)$ which has order 48 and is solvable finite
generalized triangle group.

$$G = \langle a, b \mid a^2, b^3, (ababab^2)^2 \rangle$$

$$\mathcal{L}_G = \{a, b, bb^a, b^a b, (ba)^4\}$$

$\nu(G)$ has order 110592. (Coset enumeration or Solvable

$$\Gamma(G^{ab}) = C_4 \times C_3 \quad H_2(G) = 1 \quad H_3(G) = C_2 \times C_2 \\ J_2(G) = C_2.$$

$$G \otimes G = C_2 \times SL(2, 3) \quad G \wedge G = SL(2, 3).$$

A simple example (cont)

$$\begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & & \downarrow & \\
 C_2 \times C_{24} & \longrightarrow & C_4 \times C_3 & \xrightarrow{\psi} & C_2 & \longrightarrow & \\
 \parallel & & \parallel & & \downarrow & & \\
 C_2 \times C_{24} & \longrightarrow & C_4 \times C_3 & \xrightarrow{\psi} & C_2 \times SL(2, 3) & \longrightarrow & \\
 & & & & \kappa \downarrow & & \\
 & & & & SL(2, 3) & \longrightarrow & \\
 & & & & \downarrow & & \\
 & & & & 1 & &
 \end{array}$$

Finitely Presented Groups

Proposition 8. *If G is a finitely generated group then $H_3(G)$ is a finitely generated abelian group.*

Proposition 9. *If G is a finitely generated group then $H_2(G)$ is a finitely generated abelian group.*

The two propositions below follow from the following exact sequences

$$\begin{array}{ccccccc} H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \longrightarrow & J_2(G) & \longrightarrow & H_2(G) \\ & & & & & & \\ 0 & \longrightarrow & J_2(G) & \longrightarrow & G \otimes G & \longrightarrow & G' \longrightarrow \end{array}$$

Proposition 10. *If G is a finitely presented group then $H_2(G)$ is a finitely generated abelian group.*

Proposition 11. *If G is a finitely presented group and $H_2(G)$ is finitely generated then $G \otimes G$ and $G \wedge G$ are finitely generated.*

Observations

All “tensor” computations can be done as commutator computations. We truly have a commutator connection. The following is due to Rocco (1991).

Lemma 12. *The following relations hold in $\nu(G)$:*

- (i) $[g_3, g_4^\varphi][g_1, g_2^\varphi] = [g_3, g_4][g_1, g_2^\varphi]$ and
 $[g_3^\varphi, g_4][g_1, g_2^\varphi] = [g_3, g_4][g_1, g_2^\varphi]$ for all $g_1, g_2, g_3, g_4 \in G$
- (ii) $[g_1^\varphi, g_2, g_3] = [g_1, g_2, g_3^\varphi] = [g_1^\varphi, g_2, g_3^\varphi]$ and
 $[g_1, g_2^\varphi, g_3] = [g_1^\varphi, g_2^\varphi, g_3] = [g_1, g_2^\varphi, g_3^\varphi]$ for all $g_1, g_2, g_3 \in G$
- (iii) $[g, g^\varphi]$ is central in $\nu(G)$ for all $g \in G$;
- (iv) $[g_1, g_2^\varphi][g_2, g_1^\varphi]$ is central in $\nu(G)$ for all $g_1, g_2 \in G$;
- (v) $[g, g^\varphi] = 1$ for all $g \in G'$.

Observations (cont)

Lemma 13. *Let a, b and x be elements of G such that $[x, a] = 1 = [x, b]$. Then in $\nu(G)$,*

$$[a, b, x^\varphi] = 1 = [[a, b]^\varphi, x].$$

Lemma 14. *Let x and y be elements of G such that $[x, y] = 1$. Then in $\nu(G)$,*

- (i) $[x^n, y^\varphi] = [x, y^\varphi]^n = [x, (y^\varphi)^n]$ for all integers n ;
- (ii) *If x and y are torsion elements of orders $o(x)$ and $o(y)$ then the order of $[x, y^\varphi]$ in $\nu(G)$ divides the greatest common divisor of $o(x)$ and $o(y)$.*

We have nilpotency and solvability bounds on $\nu(G)$ which make these computations easier when G is nilpotent or

Application

As an application of the earlier theoretical results using from computational results for small rank we compute the nonabelian tensor square of the free nilpotent groups of finite rank.

All calculations involve working with the commutators in the free nilpotent class 3 rank n . The problem involves finding exact structure of $[G, G^\varphi]$. Using crossed pairings, the “simple” free 2-Engel case of rank 2 took 2 published papers totaling 33 pages plus an 117 page dissertation + 42 pages of supporting materials.

The free nilpotent class 3 rank n will be written up in a published paper which includes all details.

Free nilpotent groups of class 3

Theorem 15. *Let G be a free nilpotent group of class 3 and rank n . Then $G \otimes G \cong N \times A$ where N is nilpotent of class 2 and rank $n(n-1)$ and A is free abelian of rank $f(n)$ where*

$$\begin{aligned} f(n) &= n + 2 \binom{n}{3} + 2 \binom{n}{2} + 6 \binom{n}{3} + 3 \binom{n}{4} + 3 \binom{n}{5} \\ &= n + 5 \binom{n}{2} + 8 \binom{n}{3} + 3 \binom{n}{4} \\ &= \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}. \end{aligned}$$

Capable Groups

Another application of computing the nonabelian tensor group is determining if a group is capable.

A group G is **capable** if there exists a group H such that $G \cong H/Z(H)$.

The epicenter $Z^*(G)$ of a group G is the intersection of central extensions of G .

Theorem 16 (Beyl, Felgner, Schmid). *A group G is capable if and only if $Z^*(G) = 1$.*

Exterior Square and Center of a group

Let $\nabla(G)$ denote the central subgroup of $G \otimes G$ generated by elements $x \otimes x$ for $x \in G$.

Then $G \wedge G = (G \otimes G) / \nabla(G)$.

We define the exterior center of a group G as

$$Z^\wedge(G) = \{a \in G \mid a \wedge g = 1_\wedge, \forall g \in G\}.$$

$Z^\wedge(G)$ is subgroup of the center of G .

Theorem 17 (Ellis). *For any group G , the epicenter of G coincides with the exterior center, i.e. $Z^*(G) = Z^\wedge(G)$. Hence a group is epicenter capable if and only if $Z^\wedge(G) = 1_G$.*

Determining Capability

Let T be the subgroup of $[G, G^\varphi]$ generated by $[g, g^\varphi]$. T is isomorphic to $\nabla(G)$. Hence $[G, G^\varphi]/T$ is isomorphic to $\mathcal{C}(G)$.

We note that T is central in $\nu(G)$.

Let $\tau(G) = \nu(G)/T$. Then $[G, G^\varphi]$ in $\tau(G)$ is isomorphic to $\mathcal{C}(G)$.

Proposition 18. *For any group G , the epicenter of G is the intersection of the image of G in $\tau(G)$ with $Z(\tau(G))$.*