1. Define the function

\[ f(x) = \begin{cases} 
    x & \text{if } x \in [0, 1] \text{ is rational}, \\
    0 & \text{if } x \in [0, 1] \text{ is irrational}.
\end{cases} \]

Prove that \( \int_a^b f = 0 \) and \( \int_a^b f \geq \frac{1}{2} \). [7 pts]

2. Suppose that the function \( f : [a, b] \to \mathbb{R} \) is monotonically decreasing and let \( P_n \) be the regular partition of \([a, b]\) into \( n \) intervals of length \((b - a)/n\).

   a. Show that \( U(f, P_n) - L(f, P_n) = (f(a) - f(b))(b - a)n \). [7 pts]

   b. Use part (a) and the Archimedes-Riemann Theorem to show that \( f \) is integrable on \([a, b]\). [3 pts]

3. a. For a partition \( P = \{x_0, x_1, \ldots, x_n\} \) of the interval \([a, b]\), show that

\[ \sum_{i=1}^{n} [x_i - x_{i-1}]^2 \leq [b - a] \cdot \text{gap } P. \] [6 pts]

   b. Suppose that the function \( f : [a, b] \to \mathbb{R} \) is Lipschitz; that is, there exists a constant \( c \geq 0 \) such that

\[ |f(u) - f(v)| \leq c|u - v|, \quad \text{for all } u, v \in \mathbb{R}. \]

For a partition \( P \) of \([a, b]\), prove that

\[ 0 \leq U(f, P) - L(f, P) \leq c[b - a] \cdot \text{gap } P. \]

Hint: Use the Extreme value Theorem and the estimate from (a). [6 pts]
c. Use the Darboux sum difference estimate from part (b) and the Archimedes-Riemann Theorem to show that a Lipschitz function is integrable. [6 pts]

4. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of nonnegative numbers. Show that if

\[
\lim_{n \to \infty} [a_n + b_n] = 0, \quad \text{then} \quad \lim_{n \to \infty} a_n = 0 \quad \text{and} \quad \lim_{n \to \infty} b_n = 0. \quad [7 \text{ pts}]
\]

5. Suppose the continuous function \( f: [a, b] \to \mathbb{R} \) has the property that

\[
\int_{c}^{d} f \leq 0, \quad \text{whenever} \quad a \leq c < d \leq b.
\]

Prove that \( f(x) \leq 0 \) for all \( x \in [a, b] \). Is this true if we only require that \( f \) by integrable? [8 pts]