This assignment is worth 50 points, distributed as indicated. Be sure to justify your claims and explain your reasoning. This assignment is due on **Wednesday, April 8**.

1. a. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) has the property that
\[-x^2 \leq f(x) \leq x^2 \text{ for all } x.\]
Prove that \( f \) is differentiable at \( x = 0 \) and that \( f'(0) = 0 \). \([5 \text{ pts}]\)

b. Let \( g(x) = x^2 \sin(1/x) \) for \( x \neq 0 \) and \( g(0) = 0 \). Use theorems 4.6 and 4.14 to show that \( g \) is differentiable at \( x_0 \neq 0 \) and calculate \( g'(x_0) \). You may use the fact that the derivative of sine is cosine. \([2 \text{ pts}]\)

c. Show that \( g \) is differentiable at \( x = 0 \) and that \( g'(0) = 0 \). \([3 \text{ pts}]\)

d. Show that \( g' \) is not continuous at \( x = 0 \). \([4 \text{ pts}]\)

2. A function \( f : \mathbb{R} \to \mathbb{R} \) is called **even** if \( f(x) = f(-x) \) for all \( x \); \( f \) is called **odd** if \( f(x) = -f(-x) \) for all \( x \). Prove that if \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and odd, then \( f' \) is even. \([6 \text{ pts}]\)

3. Let \( f(x) = x^2 \sin(1/x) \) for \( x \neq 0 \), \( f(0) = 0 \) and \( g(x) = x \) for \( x \in \mathbb{R} \).

a. Calculate \( f(x) \) for \( x = \frac{1}{\pi n} \), \( n = \pm 1, \pm 2, \ldots \). \([2 \text{ pts}]\)

b. Explain why
\[
\lim_{x \to 0} \frac{g(f(x)) - g(f(0))}{f(x) - f(0)}
\]
is meaningless. Observe that this composition requires the use of the “fix” \( h(y) \) in the proof of the Chain Rule. \([4 \text{ pts}]\)

4. Prove that \(|\cos u - \cos v| \leq |u - v| \) for all \( u, v \in \mathbb{R} \). \([6 \text{ pts}]\)
5. a. Let $I$ be an open interval. Suppose that the function $f : I \to \mathbb{R}$ is continuous and that at the point $x_0 \in I$, $f(x_0) > 0$. Prove that there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $f(x) > 0$. [6 pts]

b. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, that $f' : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$ and that $f'(0) > 0$. Prove that there is an open interval $I$ containing 0 such that $f : I \to \mathbb{R}$ is strictly monotone. [6 pts]

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a positive number $c$ such that

$$f'(x) \geq c \quad \text{for all } x \in \mathbb{R}.$$ 

Prove that

$$f(x) \geq f(0) + cx, \quad \text{if } x \geq 0 \quad \text{and} \quad f(x) \leq f(0) + cx, \quad \text{if } x \leq 0.$$ 

Use these inequalities to prove that $f(\mathbb{R}) = \mathbb{R}$. [6 pts]