1. 1296 has a prime factorization of $2^43^4$. By the Fundamental Theorem of Arithmetic $x = y = 4$. So $x + y = 8$.

2. The possible factorings of 2005 are $1 \cdot 2005$ or $5 \cdot 401$. Since $A, C$, and $M$ are all digits we have that $A + M + C \leq 3 \cdot 9 = 27$ and $100 \cdot A + 10 \cdot M + C \leq 900 + 90 + 9 = 999$. Therefore, the only possibility is $100 \cdot A + 10 \cdot M + C = 401$ and $A + M + C = 5$ and $A$ will be the hundreds digit of 401. So $A = 4$.

3. Using the laws for logarithms we can simplify the equation to $\log_{10}(2^a3^b5^c7^d) = 2005$. Which is equivalent to $2^a3^b5^c7^d = 10^{2005} = 2^{2005}5^{2005}$. So we must have $a = 2005$, $b = 0$, $c = 2005$ and $d = 0$. Therefore, there is only one solution. (I should mention that the Fundamental Theorem of Arithmetic is only stated for nonnegative integer exponents, but it is not hard to show that the unique factorization result is true for rational exponents.)

4. $2^{2004}5^{2006} = 10^{2004} \cdot 5^2 = 25 \cdot 10^{2004}$. The $10^{2004}$ will not contribute to the digits. So the sum of the digits will be $2 + 5 = 7$.

5. This is very similar to the previous example. $25^{64} \cdot 64^{25} = (25^{32} \cdot 8^{25})^2$ giving us $N = 25^{32} \cdot 8^{25}$. Therefore, $N = 5^{64} \cdot 2^{75} = 10^{64} \cdot 2^{11}$. Again the $10^{64}$ will not contribute any digits. $2^{11} = 2048$, so the sum of the digits is 14.

6. We are given that $x^2 - 63x + k = 0$ has two prime roots. Call them $p$ and $q$. So $x^2 - 63x + k = (x - p)(x - q) = x^2 - (p + q)x + pq$. So we just need to count the number of ways that two primes can add up to 63. $2 + 61 = 63$ is the only possibility. So the correct answer is 1. (This is not hard to check since we need to find out how many primes $p$ and $q$ exist such that $p = 63 - q$. If $q$ is odd then $p$ is even and we can conclude that $p$ is 2 and $q$ is 61.).

7. As we noted in class the number of distinct divisors of a natural number $n$ with prime factorization $p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$ is $(n_1 + 1)(n_2 + 2) \cdots (n_k + 1)$. If this number is odd, then $n_1, n_2, ..., n_k$ must all be even. This means that $n$ is a perfect square and the only perfect squares less than 50 are 1, 4, 9, 16, 25, 36, 49. So the final answer is 7.

8. Let $x = 0.\overline{ab}$. Then $99x = 100x - x = ab.\overline{ab} - 0.\overline{ab} = ab$. Therefore, $x = \frac{ab}{99}$. So the possible denominators in lowest terms are all divisors of 99, but since $a$ and $b$ are not both 0 or 9 we can’t have 1 as the denominator. Therefore the possible denominators are 3, 9, 11, 33 and 99. So there are 5 different possibilities.

9. Note that
$$1, 005, 010, 010, 005, 001 = 1, 000, 000, 000, 000, 000 + 5 \times 1, 000, 000, 000, 000 + 10 \times 1, 000, 000, 000 + 10 \times 1, 000, 000 + 5 \times 1000 + 1$$
Which is $1 \times 10^15 + 5 \times 10^12 + 10 \times 10^9 + 10 \times 10^6 + 5 \times 10^3 + 1$ or
$$1000^5 + 5 \cdot 1000^4 + 10 \cdot 1000^3 + 1 \cdot 1000^2 + 5 \cdot 1000 + 1.$$ By the binomial theorem one can recognize this sum as $(1000 + 1)^5 = 1001^5$. $1001 = 7 \cdot 11 \cdot 13$. So $1, 005, 010, 010, 005, 001 = 7^5 \cdot 11^5 \cdot 13^5$.

10. $n^2 - m^2 = (n - m)(n + m)$ and the factorings of 2008 are $1 \cdot 2008$, $2 \cdot 1004$, $4 \cdot 502$ and $8 \cdot 251$. We must have $n - m$ the smaller of the two factors. So we need to solve the following pairs of equations and see which ones give integer solutions. $n - m = 1$ and $n + m = 2008$, $n - m = 2$ and $n + m = 1004$, $n - m = 4$ and $n + m = 502$, $n - m = 8$ and $n + m = 251$. The only solutions we get are $(503, 501)$.
11. Let $a, b, c, d, e$ be the integers written in an order so that $a \leq b \leq c \leq d \leq e$. The hypothesis is that $abcde = a + b + c + d + e$. It is easy to see that $a + b + c + d + e \leq 5e$. Therefore, $abcde \leq 5e$. So we can conclude that $abcd \leq 5$. The possibilities for $\{a, b, c, d\}$ are $\{1, 1, 1, 5\}$, $\{1, 1, 1, 4\}$, $\{1, 1, 1, 3\}$, $\{1, 1, 1, 2\}$ and $\{1, 1, 2, 2\}$. The first set would give $5e = 8 + e$ which means $e = 2$, but $e \geq d = 5$ which is a contradiction. For the other sets we make similar arguments and the only sets that work are $\{1, 1, 1, 3, 3\}$, $\{1, 1, 1, 2, 5\}$ and $\{1, 1, 2, 2, 2\}$.

12. If $n = p^2 - q^2$ with $p - q > 1$, then $p^2 - q^2 = (p - q)(p + q)$ is composite since $p - q$ and $p + q$ are integers greater than 1.

In the other direction suppose that $n$ is composite. We have two cases.

(Case 1) $n$ is perfect square then $n = p^2 = p^2 - 0^0$ and we are done.

(Case 2) $n$ is an odd composite and not a perfect square. Then we can write $n$ as $s \cdot t$ with $s > t > 1$ and $s, t$ are both odd. This means both $s + t$ and $s - t$ are both even and it is easy to check that $n = \left(\left(\frac{s + t}{2}\right)^2 - \left(\frac{s - t}{2}\right)^2\right)$. So we are done.