The development of the irrational numbers

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The Budding Intuitionalist

"Obviously the real numbers are complete. It's just the way they are!"

—Sarah, a student in real analysis

History: Arithmetical Constructions

1858 Richard Dedekind presents his construction of the real numbers while lecturing at Zürich Polytechnic.

1863–4 Karl Weierstrass presents his construction while lecturing on analytic functions in Berlin.

1869 Charles Méray is the first to publish his construction (developed while lecturing on differential calculus).

1872 Dedekind, G. Cantor and H. Heine publish their approaches. E. Kossak publishes notes from Weierstrass' lectures, including a construction of the real numbers. (Weierstrass never published his construction.) Méray publishes a second paper.

Dedekind Cuts

Definition. *A* ***cut*** *(in the rational numbers) is a decomposition of* $\mathbb{Q}$ *into two disjoint sets* A_1 *and* A_2 *such that* $a_1 < a_2$ *for all* $a_1 \in A_1$ *and* $a_2 \in A_2$*.*

- For a cut (A_1, A_2), if either A_1 has a maximal member or A_2 a minimal member, $a \in \mathbb{Q}$, then the cut is said to be *produced by* a. The two cuts produced by a are considered to be the same.
- If this is not the case, then a new (irrational) number α is *created*, which is considered to be completely determined by the cut.
- The set $\mathbb{R}$ of real numbers is created by associating each cut with the number producing it.

From Dedekind, *Stetigkeit und irrationale Zahlen*, 1872.

Weierstrass' Construction: Definitions

- Given an positive integer n, the **fractional part** of n is the expression $1/n$.
- A collection of fractional parts, either finite or infinite, will be referred to as an **aggregate**.
- When an aggregate contains a finite number of terms, it is regarded as equal to its sum.

Transformations of Aggregates

Comparisons between two aggregates are made using the following transformations:

1. n elements of the form $1/n$ may be replaced by 1; similarly k fractional parts of the form $1/kl$ may be replaced by $1/l$.

2. Any number can be replaced by its fractional parts (e.g., 1 could be replaced by $n(1/n)$).

Order and Equality

- Let a and b be two aggregates. We say that $a \leq b$ if and only if, for all proper subaggregates a' of a that contain finitely many fractional parts, we can transform a' to a'', so that every fractional part in a'' occurs in b, including any multiplicities.

- We say $a = b$ if and only if $a \leq b$ and $b \leq a$.

Note: Equality is an equivalence relation.

The Real Numbers

- An aggregate a has **finite value** if and only if there exists an aggregate b, containing finitely many members, such that $a \leq b$.

- A **positive real number** is an equivalence class of aggregates which has finite value.

- A real number is rational if and only if there is an aggregate with finitely many elements in its equivalence class.

Méray: "Convergent Variants"

Definition. *A* **variant** *(*une quantité variable*) v is an infinite sequence of rational numbers $v_1, v_2, v_3, \ldots$ The variant is said to be* **convergent** *if it satisfies the Cauchy criterion: for any $\epsilon > 0$ there exists a natural number n so that $|v_{n+p} - v_n| < \epsilon$, for any natural number p.*

- If there exists a rational number V such that $V - v_n$ tends to zero, then V is the limit of the convergent variant v.

- If v is a convergent variant for which no such rational number exists, we say that v has a **fictitious limit** (*une limite ficitve*) V.

The Real Numbers: Order and Equality

Definition. *The* ***real numbers*** *are the rational numbers together with the fictitious limits.*

- Given any two real numbers U and V, we say they are **equivalent** when the difference $u_m - v_n$ tends to zero.

- U is said to be **greater (less) than** V, if $u_m - v_n$ is always positive (negative) for sufficient large n and m.

From Méray, "Remarques sur la nature des quantités définies par la condition de servir de limites á des variables données," *Revue de Sociétés savants, sci, math, phys, nat,* **2** no. 4 pp. 280–289, 1869.

Cantor and Heine

- Cantor and Heine were colleagues at Halle at the time of their publications (1872).
- Developed their theory independently of Méray.
- Differences among the three are philosophical.
  - Méray: Fictitious limits (*limites fictives*).
  - Cantor: "Associates" a real number to each equivalence class of Cauchy sequences (*Fundamentalreihen*).
  - Heine: The real numbers are signs (*greifbare Zeichen*) for each equivalence class of Cauchy sequences.

In the Classroom

- How would you represent π using the different construction methods? How many different representations can you find?

- Using the definitions of order given with each construction, can you show that $\sqrt{2} < e$?

- Do the different constructions produce the same real numbers? Can you prove they are equivalent?

- How would you define the arithmetic operations on the different constructions?

The Last Word

"Why are the easy things so hard to prove?"

–Sarah

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