INCONICS THROUGH TWO GIVEN POINTS

An inconic of a triangle is a conic tangent to the sidelines of the triangle.

Suppose three distinct, not concurrent lines and two distinct points are given. Let ABC be the triangle bounded by these lines. If the points are U=u:v:w and X=x:y:z (normalized barycentrics or exact trilinears with respect to ABC)(*), then the perspectors Q of the inconics of ABC through U, V are given by:

$$Q(\{u,v,w\},\{x,y,z\}) = \frac{1}{F(\{u,v,w\},\{x,y,z\},\{\beta,\gamma\})} : \frac{1}{G(\{u,v,w\},\{x,y,z\},\{\beta,\gamma\})} : \frac{1}{G(\{u,w,v\},\{x,z,y\},\{\gamma,\beta\})}$$

where:

$$F(\{u,v,w\},\{x,y,z\},\{\beta,\gamma\}) = (vz-wy)\left(x(vz-wy)+w(uv-xy)\beta - v(uw-xz)\gamma\right)$$

$$\begin{aligned} G(\{u,v,w\},\{x,y,z\},\{\beta,\gamma\}) = \\ &((vz+wy)(uz+wx)-2wz(uv+xy))x \\ &+((uz+wx)(uv+xy)-2ux(vz+wy))w\beta \\ &+\left((w-z)^2ux+(u-x)^2wz\right)v\gamma \\ &-2(w-z)(u-x)uvw\beta\gamma \end{aligned}$$

$$\beta = \pm\sqrt{\frac{xz}{uw}} \quad \gamma = \pm\sqrt{\frac{xy}{uv}}$$

Therefore, there are four inconics of ABC through the given points, obtained from the combination of the distinct values of β and γ.

If U, X are both triangle centers or if they are a bicentric pair, just one of the four inconics is a central inconic (conjectured), i.e., an inconic with center and perspector being triangle centers. We must try all the combinations of β and γ in order to find which one gives a central perspector and a central inconic, because this depends on the relative position of points U and X.

(*): The given points must both lie in the same side with respect to every tangent line. This means that both points must be interior or exterior to ABC.

Equations obtained by César Eliud Lozada from the method by:

Eagles, Thomas Henry, Constructive Geometry of Plane Curves, McMillan and Co., London, 1885, problem 83, pp.134 and problem 112, pp. 182. (Available here).

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