Triangle centres around Vecten circles

Stanisław Majchrzak

June 21, 2021

Abstract

In this paper I will present another construction of points X_{39387} and X_{39388}. The first section is a short note about Vecten triangles and Vecten points. In the second I will introduce useful notation. The third section includes the new properties and their proofs.

1 Introduction

Let ABC be a triangle. On the sides BC, CA, AB of the triangle, construct outwardly drawn three squares. The centres of these squares form a triangle that is perspective to triangle ABC. This triangle is called Outer Vecten triangle of triangle ABC and their prospector is called Outer Vecten point (listed at Encyclopedia of Triangle Centers as X_{485}). Similarly if we construct the squares inwardly, the corresponding centres of these squares form Inner Vecten triangle that is perspective to triangle ABC with Inner Vecten point as a prospector (listed as X_{486} at Encyclopedia of Triangle Centers). The circumcircle of Outer Vecten triangle is called Outer Vecten circle and the circumcircle of Inner Vecten triangle is called Inner Vecten circle.

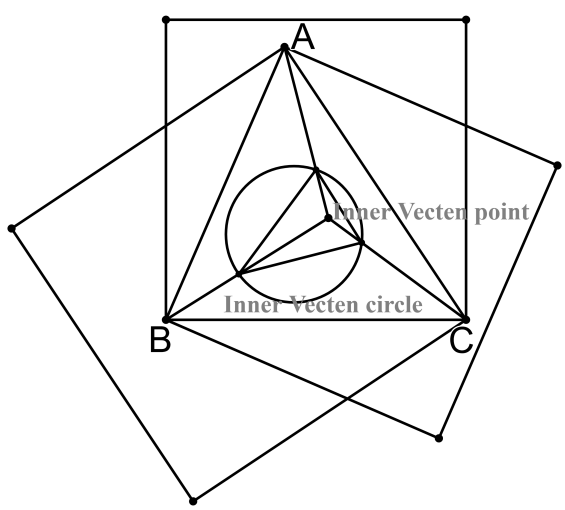

Figure 1

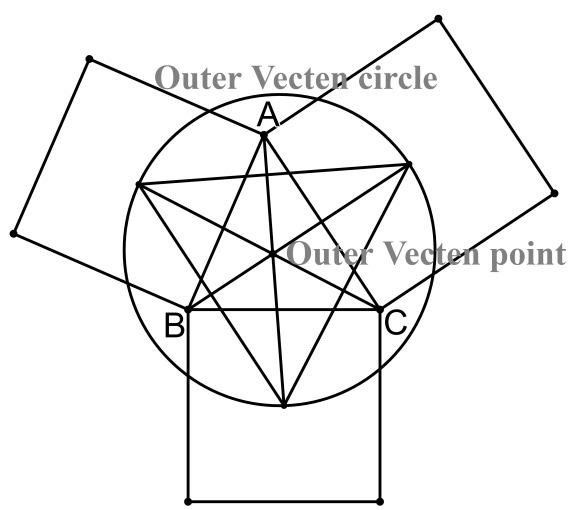

Figure 2

2 Notation

In triangle ABC we will use the following notation:

- H - the orthocentre of triangle ABC
- $V_aV_bV_c$ - Inner Vecten triangle of triangle ABC
- $V_AV_BV_C$ - Outer Vecten triangle of triangle ABC
- X_n - n-th triangle centre listed at [1]

We denote a projection of point X to line k by X^k.

For two points X and Y by ω_{XY} we denote the circle with diameter XY.

3 Triangle centres

Theorem 3.1. *Triangle ABC and its Vecten triangles have common centroid.*

The proof of this theorem includes the following lemma:

Lemma 3.1. *Medial triangle of triangle ABC is a Vecten triangle of a Vecten triangle of ABC.*

Proof. Let M be the midpoint of segment BC. By three rotations theorem in hexagons AV_cBMCV_b and AV_CBMCV_B we have that V_bMV_c and V_BMV_C are isosceles right-angled triangles. Analogously other midpoints follow this condition as desired. □

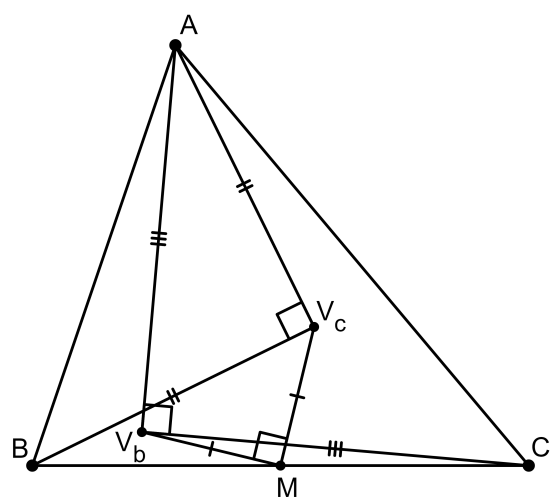

Figure 3

Proof of the theorem. Applying lemma 3.1 to triangles $V_aV_bV_c$ and $V_AV_BV_C$ we get that medial triangles of these triangles are Vecten triangles of medial triangle of triangle ABC. Thus homothety which takes triangle ABC to its medial triangle, takes its Vecten triangles to their medial triangles (because they are Vecten triangles of medial triangle of ABC). Since homothety which takes a triangle to its medial one is centred at the centroid of this triangle, it must be common for all these triangles. □

Theorem 3.2. *Let the Vecten circle of triangle ABC intersect circles ω_{BC}, ω_{CA}, ω_{AB} for the second time at points X_a, X_b, X_c respectively. Then lines AX_a, BX_b, CX_c are concurrent.*

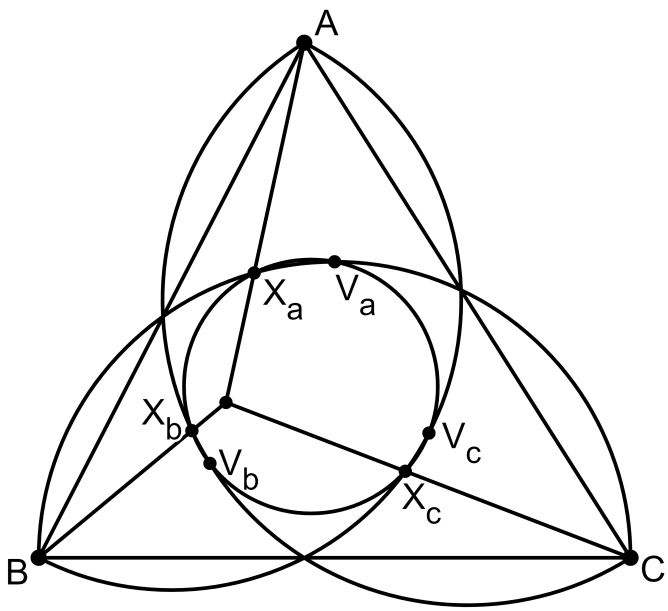

Figure 4

We present the proof of this theorem in particular steps

Lemma 3.2. *Let V be a point lying on ω_{BC} in a triangle ABC. A line passing through V perpendicular to AV meets lines BH and CH at P_B and P_C respectively. Then the following conditions are true*

1. *triangles P_BAP_C and BVC are similar*
2. *triangles BP_BA and CAP_C are similar*
3. *a line passing through B parallel to AP_C, a line passing through C parallel to AP_B and line P_BP_C are concurrent*

Proof. (1) Note that quadrilaterals $AP_BB^{AC}V$ and $AP_CC^{AB}V$ are cyclic. Hence

$$\angle VBC = \angle VB^{AC}A = \angle VP_BA \wedge \angle VCB = \angle VC^{AB}A = \angle VP_CA$$

(2) Note that $\angle ABP_B = \angle ACP_C = 90^\circ - \angle BAC$. Moreover $\angle AP_BP_C = \angle VAP_C$. Thus

$$\angle P_CAC = \angle P_CAV + \angle VAC = \angle AP_BP_C + \angle VP_BB = \angle AP_BB$$

(3) Let ω_{BC} meet line P_BP_C for the second time at U. Then

$$\angle P_CAB = \angle P_CVC^{AB} = \angle UVC^{AB} = \angle ABU$$

so $BU \parallel AP_C$. Similarly $CU \parallel AP_B$. $\square$

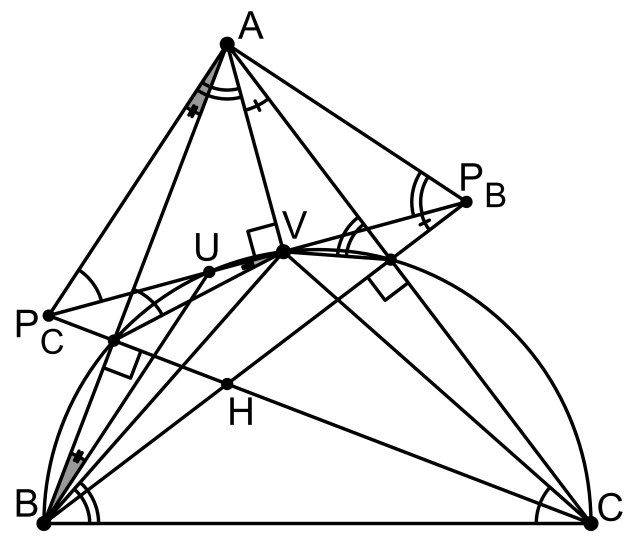

Figure 5

Lemma 3.3. *Perpendiculars at V_a, V_b, V_c to AV_a, BV_b, CV_c form a triangle whose vertices lie on lines AH, BH, CH.*

Proof. Let us denote by P_a the intersection point of perpendiculars at V_b and V_c and P_b, P_c cyclically. We will show that $P_a \in AH, P_b \in BH, P_c \in CH$. Assume that line perpendicular to AV_a passing through V_a meets rays $\overrightarrow{BH}, \overrightarrow{CH}$ at P_b', P_c' respectively. Then by lemma 3.2.1 triangle $P_b'AP_c'$ is right-angled and isosceles. Now applying lemma 3.2.2 we get that triangles $BP_b'A$ and CAP_c' are equal, so $BP_b' = AC$ and $CP_c' = AB$. Analogously perpendicular line passing through V_b passes through such point P_c'' on $\overrightarrow{CH}$ that $CP_c'' = AB$. Hence $P_c' = P_c'' = P_c$. Similarly points P_a and P_b. $\square$

Lemma 3.4. *Triangle $X_aX_bX_c$ is an orthic triangle of triangle $P_aP_bP_c$ defined in lemma 3.3.*

Proof. We mentioned that triangle P_bAP_c is an isosceles right-angled triangle. Since V_a is a projection of A to P_bP_c it must be the midpoint of segment P_bP_c. Analogously points V_b and V_c. Hence the circumcircle of triangle $V_aV_bV_c$ is a nine-point circle of triangle $P_aP_bP_c$. So it is enough to show that this circle meets circles $\omega_{BC}, \omega_{CA}, \omega_{AB}$ at points lying on lines that include sides of triangle $P_aP_bP_c$. Assume that circle ω_{BC} meets line P_bP_c at X'_a and define points X'_b, X'_c cyclically. By lemma 3.2.3 we have $BX'_a \parallel AP_c$, so $\angle P_cX'_aB = \angle AP_cX'_a = 45^\circ = \angle P_cP_aB$ which means that $X'_a \in \omega_{P_aP_c}$. Similarly points X'_b and X'_c. Thus triangle $X'_aX'_bX'_c$ is orthic triangle of $P_aP_bP_c$, so it is inscribed in nine-point circle of this triangle. Finally it gives us $X_a = X'_a$, $X_b = X'_b$, $X_c = X'_c$ as desired. $\square$

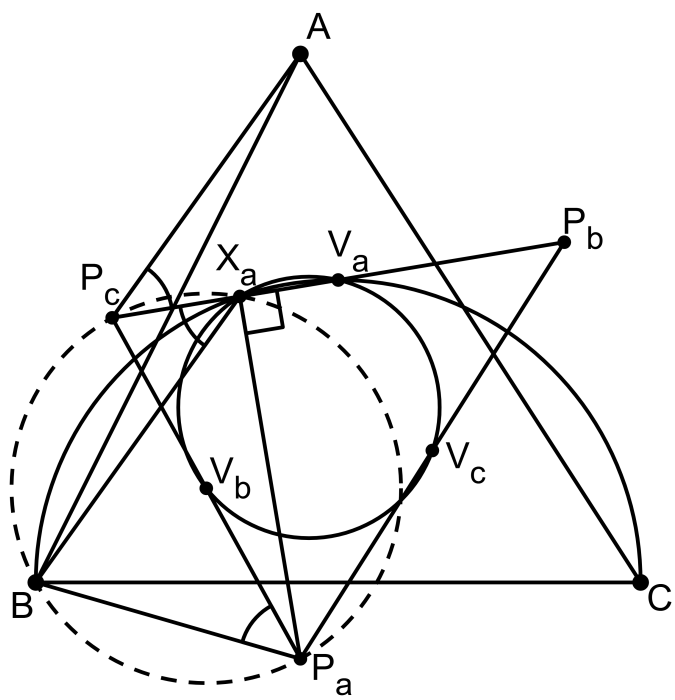

Figure 6

Lemma 3.5. *Points X and Y are isogonal with respect to $\angle(k,l)$. Assume that $K \in k$ and $L \in l$ and define $S = KY \cap LX$, $T = KX \cap LY$. Then S and T are isogonal with respect to $\angle(k,l)$.*

Proof. We know that $\frac{XX^k}{XX^l} = \frac{YY^l}{YY^k}$ and we want to show that $\frac{SS^k}{SS^l} = \frac{TT^l}{TT^k}$. By Tales theorem we have that

$$\frac{YY^l}{TT^l} = \frac{LY}{LT} \Rightarrow TT^l = YY^l \cdot \frac{LT}{LY}$$
$$\frac{XX^k}{TT^k} = \frac{KX}{KT} \Rightarrow TT^k = XX^k \cdot \frac{KT}{KX}$$
$$\frac{YY^k}{SS^k} = \frac{KY}{KS} \Rightarrow SS^k = YY^k \cdot \frac{KS}{KY}$$
$$\frac{XX^l}{SS^l} = \frac{LX}{LS} \Rightarrow SS^l = XX^l \cdot \frac{LS}{LX}$$

By applying these equalities it is enough to show that $\frac{KS}{KY} \cdot \frac{LX}{LS} = \frac{KX}{KT} \cdot \frac{LT}{LY}$.

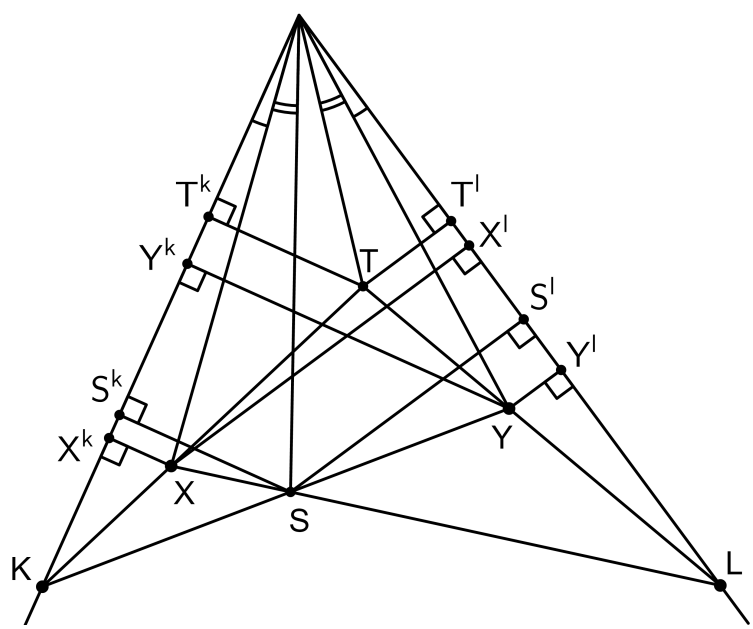

Figure 7

By Menelaus theorem applied to $\triangle KXS$ and line LT and by Menelaus theorem applied to $\triangle KTY$ and line LX we have that

$$\frac{KT}{TX} \cdot \frac{XL}{LS} \cdot \frac{SY}{YK} = \frac{KX}{XT} \cdot \frac{TL}{LY} \cdot \frac{YS}{SK}$$
$$\frac{KT}{KY} \cdot \frac{LX}{LS} = \frac{KX}{KS} \cdot \frac{LT}{LY}$$

as desired. $\square$

Lemma 3.6. *Lines V_BC^{AB} and V_CB^{AC} meet at V_a.*

Proof. Because quadrilateral $AC^{AB}CV_B$ is cyclic we have

$$\angle AC^{AB}V_B = \angle ACV_B = 45^\circ = \angle CAV_B = \angle CC^{AB}V_B = \angle CC^{AB}V_a$$

so V_a lies on V_BC^{AB} and so on. □

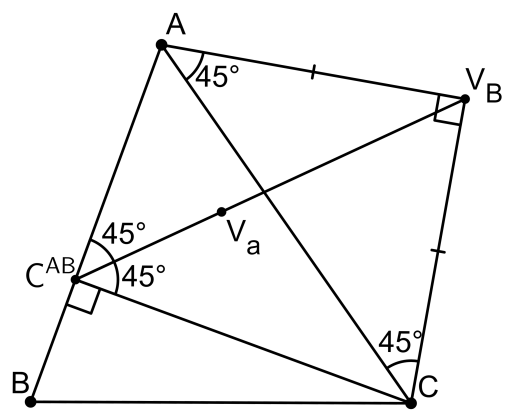

Figure 8

Lemma 3.7. *Orthic triangle of triangle ABC and Vecten triangle of triangle ABC are perspective.*

Proof. Assume by way of contradiction that lines $A^{BC}V_A$, $B^{CA}V_B$, $C^{AB}V_C$ form a triangle. Let us denote $S_A = B^{CA}V_B \cap C^{AB}V_C$ and S_B, S_C cyclically. Since points V_B and V_C are isogonal with respect to $\angle BAC$, by lemmas 3.5 and 3.6 we have that S_A and V_a are so too. Analogously for pairs S_B, V_b and S_C, V_c. Hence triangles ABC and $S_AS_BS_C$ must be perspective with isogonal conjugate of one of Vecten points as a prospector. Thus by Desarques theorem these triangles have a perspetrix. That's a contradiction because that would mean that the orthic triangle of triangle ABC is a segment. Hence $S_A = S_B = S_C$. □

Proof of the theorem. We noticed that triangles P_bAP_c, P_cBP_a, P_aCP_b are isosceles right-angled triangles. Thus triangle ABC is a Vecten triangle of triangle $P_aP_bP_c$. Lemma 3.7 completes the proof. □

The given proof in the case of Inner Vecten circle can be easily repeated for the case of Outer Vecten circle. Thus we proved the existence of two triangle centres. We will show that these are X_{39387} and X_{39388}.

Proof. We will show that a circle centred at V_b passing through B meets a circle centred at V_c passing through C at X_a. It is enough to show that V_b lies on perpendicular bisector of segment BX_a. Let M be the midpoint of segment BC. We showed that $\angle V_bMV_c = 90°$. Because M lies on perpendicular bisector of segment BX_a, it is enough to show that $MV_b \perp BX_a$, so that $MV_c \parallel BX_a$. Let us consider triangle $P_aP_bP_c$ once again. By lemma 3.2.3 we have that $BX_a \parallel AP_c$, so we will show that $AP_c \parallel MV_c$. By theorem 3.1 triangles ABC and $P_aP_bP_c$ have common centroid. Considering a homothety centred at X_2 which takes triangle ABC to its medial one we get that point A is taken to M and point P_c is taken to V_c. Hence $AP_c \parallel MV_c$ as desired. Analogously points X_b and X_c, so the proof is completed in case of Inner Vecten circle. The other case is analogous. □

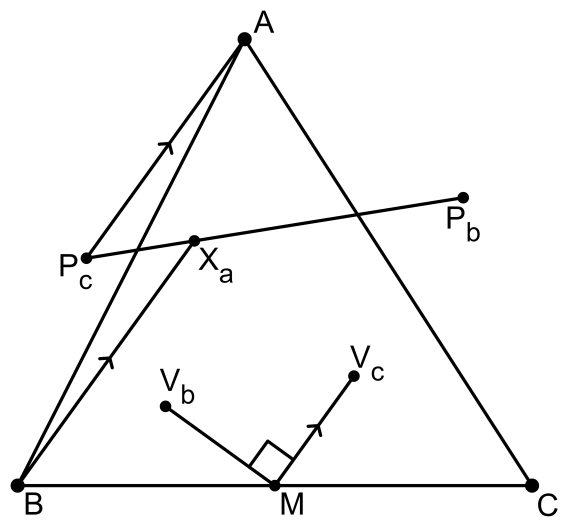

Figure 9

References

[1] C. Kimberling: *Encyclopedia of Triangle Centers*

[2] S. Majchrzak: *Od trójkatów do krzywych stożkowych, czyli o prostej innej niż wszystkie*, Tarnów 2020

[3] https://mathworld.wolfram.com/InnerVectenTriangle.html

[4] https://mathworld.wolfram.com/OuterVectenTriangle.html

[5] https://mathworld.wolfram.com/InnerVectenCircle.html

[6] https://mathworld.wolfram.com/OuterVectenCircle.html