A New Electric Field Interpretation of Barycentric and Trilinear Coordinates

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Abstract

The usual physical interpretation of barycentric coordinates is given in terms of center of mass of a set of masses placed at the vertices of the reference triangle. However, in this article, we present a new physical interpretation of barycentric and trilinear coordinates in terms of electric fields of line charges placed along the sides of the triangle. Although the proofs are not very enlightening, the results are quite useful as well as surprising.

1 Notation

ABC is the reference triangle, a, b, c represent the side-lengths opposite to the vertices A, B, C. α, β, γ represent the measure of the angles $\angle A, \angle B, \angle C$, unless specified otherwise.

The barycentric and trilinear coordinate systems are two of the most commonly used homogeneous coordinate systems in analytical geometry. These coordinate systems use triangles as references.
The barycentric coordinates $p : q : r$ (or simply barycentrics) of a point P wrt the triangle ABC is the ratio of the areas of the triangles $[BPC] : [CPA] : [APB]$ (where the areas are signed). Similarly, the trilinear coordinates (or simply trilinears) $x : y : z$ of a point P wrt the triangle ABC is the ratio of the distances of the point P from the sidelines BC, CA, AB respecticely (where the distances are signed). Thus, it is easy to see that the point with trilinears $x : y : z$ is the same as the point with barycentrics $ax : by : cz$.

2 The classical physical interpretation

The classical physical interpretation of barycentric coordinates is given in terms of center of mass of a set of point masses placed at the vertices of a triangle.

Theorem 2.1

If point masses of magnitude p, q, r are placed at the vertices of $\triangle ABC$, then the center of mass of the system is the point with barycentric coordinates $p : q : r$.

There exists no well known analogous physical interpretation of trilinear coordinates.

3 Electric field interpretation

Electric field at a point is the force that a unit positive charge would experience when placed at that point in the presence of other charges. Force is a vector quantity, and so is the electric field. The electric field due to a point charge q at the point P at a distance r from it is given by

$$E = \frac{1}{4\pi\epsilon}\frac{q}{r^2} = \frac{kq}{r^2}$$

The direction of this field is the same as the direction of the vector joining the position of the charge and the point P.

3.1 Electric field due to an infinite line of charge

The electric field due to a uniform infinite line of charge (see [1], pages 1-4) at a distance r from it with charge per unit length λ is given by

$$E = \frac{2k\lambda}{r}$$

This field is directed cylindrically outwards from the line charge.

3.2 Electric field due to a finite line of charge

Lemma 3.1

The electric field due to a finite uniform line charge AB at a point P is the same as the electric field at that point due to the part of the arc with center P tangent to AB (extended, if needed), between the rays PA and PB, where the arc and the line charge have the same charge per unit length.

Proof. Let D be the foot of perpendicular from P onto AB. Length of the segment $PD = d$. We prove that the electric field due to any infinitesimal element of the line is the same as the electric field due to the infinitesimal element of the ring obtained on projecting the element from the line onto the arc through the point P.

Assume an infinitesimal part of the line with length dx at a distance x from the point D. Upon projecting this through P onto the arc, let the infinitesimal part of the arc obtained be at an angle θ wrt PD and have an angular width $d\theta$. Now, $x = d \cdot \tan\theta$ and differentiating yields

$$dx = d \cdot \sec^2\theta \cdot d\theta$$

$$dE_{line} = \frac{k(\lambda \cdot dx)}{(d \cdot \sec\theta)^2}$$

$$dE_{arc} = \frac{k(\lambda \cdot d \cdot d\theta)}{d^2} = \frac{k(\lambda \cdot dx)}{(d \cdot \sec\theta)^2}$$

where the last equality follows from the relation between $d\theta$ and dx. The magnitude of the field due to these infinitesimal elements is the same, and so is the direction. This implies that the electric field due to the complete line charge is the same as that due to the arc (upon integrating the infinitesimal fields).

□

We state, without proof, the result for the electric field due to a uniform arc of charge at its center (having radius r, charge per unit length λ and subtending an angle θ at the center)

$$E = \frac{2k\lambda(\sin\frac{\theta}{2})}{r}$$

The field is directed along the bisector of the angle subtended by the arc at the center and is in the plane of the arc.

4 The main result

Theorem 4.1

Replace the sidelines BC, CA, AB by infinite line charges with linear charge densities (charge per unit length) p, q, r where pqr is not equal to zero. Then the electric field is zero at the point with barycentric coordinates $p : q : r$ wrt the triangle ABC.

Theorem 4.2

Replace the sides BC, CA, AB by finite line charges with linear charge densities p, q, r where pqr is not equal to zero. Then the electric field is zero at the point with trilinear coordinates $p : q : r$ wrt the triangle ABC.

In the following, we assune that p, q, r are positive, however, in the case when one of them is negative, a similar proof with minor modifications works (two of p, q, r being negative is the same as one of them being negative, and all three of p, q, r being negative is the same as none of them being negative).

Proof. In both cases, let the point(s) where the electric field is zero be P, and the distances of P from BC, CA, AB be d_1, d_2, d_3 respectively.

For infinite line charges

The fields due to individual line charges are perpendicular to the sides. The field due to the infinite line charges BC, CA, AB are $\frac{2kp}{d_1}$, $\frac{2kq}{d_2}$, $\frac{2kr}{d_3}$. Now, by Lami's theorem (see [2]), we get

$$\frac{2kp}{d_1 \cdot \sin(\pi - \alpha)} = \frac{2kq}{d_2 \cdot \sin(\pi - \beta)} = \frac{2kr}{d_3 \cdot \sin(\pi - \gamma)}$$

Using sine rule in $\triangle ABC$, we get

$$\frac{p}{d_1 \cdot a} = \frac{q}{d_2 \cdot b} = \frac{r}{d_3 \cdot c}$$

which simplifies to $\triangle[BPC] : \triangle[CPA] : \triangle[APB] = p : q : r$, which means that the barycentrics for P are $p : q : r$.

For finite line charges

Here we use lemma 2.1. Let the measures of the angles $\angle BPC, \angle CPA, \angle APB$ be $\theta_1, \theta_2, \theta_3$. The electric fields due to the charged segments BC, CA, AB respectively at P are given by $\frac{2kp(\sin\frac{\theta_1}{2})}{d_1}$, $\frac{2kq(\sin\frac{\theta_2}{2})}{d_2}$, $\frac{2kr(\sin\frac{\theta_3}{2})}{d_3}$ These fields are directed along the angle bisectors of $\angle BPC, \angle CPA, \angle APB$ respectively. Again, Lami's Theorem yields

$$\frac{p\sin\left(\frac{\theta_1}{2}\right)}{d_1 \sin\left(\frac{\theta_2+\theta_3}{2}\right)} = \frac{q\sin\left(\frac{\theta_2}{2}\right)}{d_2 \sin\left(\frac{\theta_3+\theta_1}{2}\right)} = \frac{r\sin\left(\frac{\theta_3}{2}\right)}{d_3 \sin\left(\frac{\theta_1+\theta_2}{2}\right)}$$

But $\theta_1 + \theta_2 + \theta_3 = 2\pi$, thus, the above expression simplifies to

$$\frac{p}{d_1} = \frac{q}{d_2} = \frac{r}{d_3}$$

or $d_1 : d_2 : d_3 = p : q : r$, which means that the trilinears for P are $p : q : r$. $\square$

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